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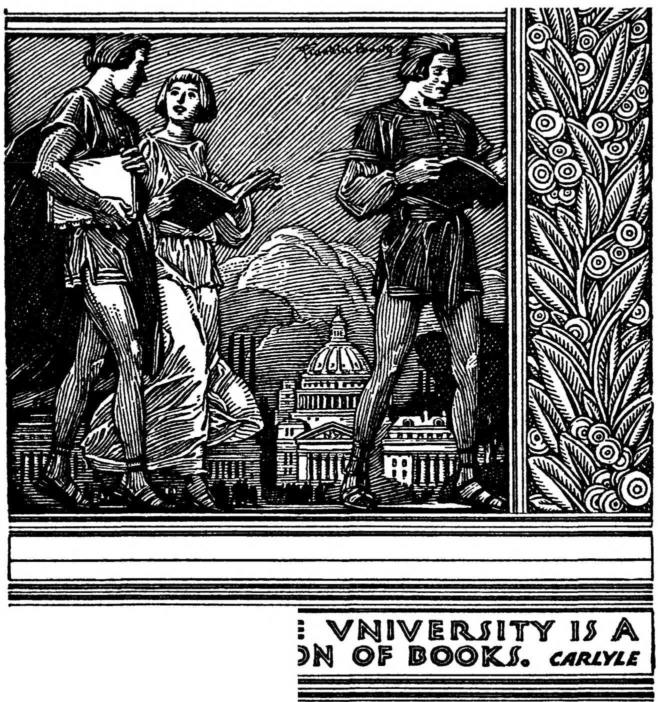

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TECHNICAL MECHANICS

STATICS, KINEMATICS, KINETICS

BY

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PREFACE

The following paragraph is an adaptation from the preface of the first edition of this work, published twenty-one years ago; it applies to the present edition.

This book might be fairly described as a theoretical mechanics for students of engineering. It is not comparable to books commonly called Theoretical Mechanics, generally intended for students of mathematics or physics; nor to books commonly titled Applied Mechanics, which generally include a treatment of strength of materials, hydraulics, etc., for students of engineering The title Technical Mechanics seems appropriate for this book; and inasmuch as it is not otherwise used in this country, it was adopted. On the theoretical side, practically every subject discussed herein has a direct bearing on some engineering problem. The applications were selected and presented for the purpose of illustrating principles of mechanics and for training students in the use of such principles, — not to furnish information, except incidentally, about the structure, machine, or what not to which the application was made.

Ten years after its first issue, this book was given a revision which resulted in a practically rewritten book. Considerable changes were made in the form, chief of which was a reduction in the number of articles, from three hundred and thirty-six to fifty-eight, though the total amount of subject matter was not greatly changed.

The book has now been rewritten again, to attain especially the following: improvements in the presentation; changes in subject matter; inclusion of a larger number of solved illustrative examples; addition of new problems; a better grading of all problems; and a return to the common practice of more minutely subdividing the matter into a relatively large number of articles.

All of Statics except Art. 82 and 93 and Chap. VIII may be mastered with no knowledge of mathematics beyond trigonometry. Kinematics and Kinetics presuppose a knowledge of simple calculus. Graphical methods are used freely, especially in Statics. The arrangement of the text is such that considerable portions may readily be selected for omission, in such wise that the remainder will constitute an independent briefer course. Thus, for example, the following might be omitted: In Statics, Art. 34 to 41, inclusive; Art. 48 to 50, inclusive; Art. 59; Art. 79 to 84, inclusive; Art. 93 and 94; and Chap. VII. In Kinematics, Art.

116, 117, 120, 121; sections 3, 4 and 5 of Chap. IX. In Kinetics, sections 3 and 4 of Chap. XI; section 5 of Chap. XII; and Chap. XIII.

The authors take pleasure in thanking Colonel C. C. Carter of the United States Military Academy, Professor C. H. Burnside of Columbia University, and their colleagues Professor M. O. Withey, Professor J. B. Kommers and Mr. C. A. Wiepking, for helpful suggestions and criticisms, and Mr. C. L. Neumeister for assistance in the reading of proof. They thank also *American Machinist* and *Engineering News-Record* for permission to copy and for gifts of cuts; and individuals and other journals named in the text for similar favors.

December, 1924.

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INTRODUCTION

It is the purpose of this introduction to give the student some idea as to the nature of Mechanics, — to tell him what it is about, and to indicate its uses in Engineering.

No mere definition can adequately tell what Mechanics is, nor can any brief statement do more than suggest its scope and applications, but at least an essentially correct point of view may be given by the statements that Mechanics is the science that treats of the effect of forces on bodies, and that its use is principally in the field of design of structures and machines.

It is expedient to distinguish Theoretical Mechanics and Applied Mechanics. The former comprises the body of exact laws and principles that have been mathematically deduced from certain fundamental facts; the latter consists essentially of the application of these laws and principles to what may generally be termed practical problems. Theoretical Mechanics is divided into Statics, which treats of the relations between forces that act on bodies at rest; Kinematics, which treats of motion as such — of the manner in which things move without regard to the cause of their motion; and Kinetics, which treats of the effect of forces in producing or modifying motion. Applied Mechanics is sometimes divided according to particular groups of applications; thus some texts are limited to "Mechanics of Materials," some to "Hydraulics," etc.

Further to describe the scope and use of Mechanics we sketch an example, — the design of a structure or machine to serve some definite purpose. Such design almost always involves the answering of these questions: What are the loads that come upon the structure and its parts; and how large, in what form, and of what material should these parts be made in order that they may sustain these loads (forces) with safety? Certain of the forces will be known at the outset; some can be assumed in accordance with engineering experience and judgment; and some must be computed, or solved for. Thus in the case of a bridge the weights of the vehicles it is to carry are known rather definitely; the wind pressure it must withstand can be assumed on the basis of past records and experiment; but the supporting reactions at the piers and the mutual pressures and tensions of the constituent members must be

determined. If the part under consideration is at rest (or, as we say, *in equilibrium*), the unknown forces that act on it can be determined by the principles of Statics. If the part under consideration is in motion, then the circumstances of the motion — the manner in which position and velocity change — are determined by the principles of Kinematics; and the circumstances of the motion having been ascertained, the forces that occasion it are determined by the principles of **Kinetics.**

When all the forces that act on a given part are known, there has yet to be ascertained the effect of these forces as respects the physical integrity of the part, — their effect, that is, in stretching, bending, twisting, or breaking it. The study of the relations between the forces that act on a body and the change they produce in its size and form, or the tendency they have to break it, is the province of Applied Mechanics. We are not, in this book, concerned with these effects. The scope of that part of Mechanics with which we here deal is limited to the definition and analysis of the relations existing between balanced forces, between the various aspects of pure motion, and between the forces that act on a body and the motion of that body.

The thoughtful student will not fail to perceive as he pursues the study of the subject that many of the simpler relations stated are essentially such as were already known to him through experience, though probably in a general and qualitative rather than precise and formal way. Indeed, great progress was made in the application of the principles of Mechanics long before the precise nature of these principles was understood. Experience, observation, instinctive understanding all played a part and enabled the early builders to make elaborate structures and to employ successfully mechanical devices of some complexity. But it was only with the recognition of the simple and precise principles involved that Mechanics as a science can be said to have begun. The early history of the development of that science is a record of the perception, by different thinkers or observers, and at long intervals, of principles that appear now to be simple. Often these principles were of such a nature as to appear in some commonly used mechanical device, for example, the lever and the inclined plane, the principles of which respectively were first perceived by Archimedes (287–212 B.C.) and Stevin (1548–1620). Again, they might be of a nature to be perceived from familiar occurrences or easily performed experiments; thus Galileo (1564–1642) proved by his famous experiments at the Tower of Pisa that bodies fall with equal rapidity regardless of their weight. The first broad systematization and formal statement of mechanical laws was the work of Newton (1642–1727).

"No essentially new principle (of Mechanics) has been stated since Newton's day. All that has been accomplished has been a deductive,

formal and mathematical development on the basis of his laws."[1] This development is relatively enormous, as are also the number and variety of the adaptations to engineering. Only a small part, such as is likely to be of most interest and use to engineers, is included in this book.

[1] From Mach's "Science of Mechanics." But since that (critical) work was written, a new system of Mechanics has been invented by Einstein, which in some respects supersedes the classical Mechanics, though not in the fields of engineering.

STATICS

CHAPTER I

FORCES AND FORCE SYSTEMS

1. Definition of a Force. — The word force is used in everyday speech, as in the introduction to this book, as a general term for push or pull. For most purposes it is sufficient to define a force as a push or pull exerted on one body by another; it may be more exactly defined as such an action upon a body, as, exerted alone, would result in motion of the body so acted on, or in change of motion if the body were already moving.

2. The Characteristics of a Force. — Our earliest notions about forces are based on experience with forces exerted on or by ourselves. Thus one tries to move a heavy body and notes that the force exerted has: (i) Magnitude, according to how hard one pushes or pulls; (ii) Direction, according to whether one pushes or pulls up, down, to the left or to the right; (iii) Place of application, according to where one grasps the body. These three attributes — magnitude, direction, and place of application — are sometimes called the *characteristics* of a force, and serve to describe it.

3. Units of Force. — To express the magnitude of a force, one must of course compare it to some other force regarded as a unit. Many units of force are in use; the most convenient are the so-called *gravitation units.* They are the earth-pulls on our standards for measuring quantity of material (as iron, coal, grain, sugar, etc.), commonly called standards of weight.[1] The earth-pull on any of these standards is called by the name of the standard; thus the earth-pull on the pound standard (also any equal force) is called a *pound;* the earth-pull on the kilogram standard (also any equal force) is called a *kilogram,* etc. Since the earth-pull on any given thing varies in amount as the thing is transported from place to place, gravitation units of force are not constant with regard to place. But this variation need not be regarded in most engineering calculations because any error due to such disregard is generally smaller than errors due to other approximations in the calculations. The extreme variation

[1] In common parlance the word *weight* is used in at least two senses. Thus, suppose that a dealer sells coal to a consumer by weight, and engages a teamster to deliver it by weight; to the consumer, the weight of each wagon load represents a certain amount of useful material, but to the teamster it represents a certain burden on his team due to the action of gravity on the coal. That is, weight suggests *material* to the one man and *earth-pull* to the other.

in any gravitation unit is that between its magnitudes at the highest elevation on the equator and at the poles; this difference is but 0.6 per cent. For points within the United States the extreme variation is about 0.3 per cent. We can, by specifying location, make the pound force an invariable unit. A certain value for g, the acceleration of gravity, has been adopted by the International Conference on Weights and Measures as standard. The value adopted represents what was believed to be the average value of g at sea level in latitude 45°. In accordance with this establishment of a value for "standard gravity," we may define, as the "International Standard Pound Force" the earth-pull on the pound standard at any place where gravity has a value equal to the standard value.

4. Distributed and Concentrated Forces. — The place of application of most forces with which we are concerned is a portion of the surface of the body to which the force is applied. If this portion is large in comparison with the total surface, the force is spoken of as a *distributed* force. If this portion is small in comparison with the total surface, it may, for many purposes, be regarded as a point; the force is then spoken of as a *concentrated* force, and the place of application is called the *point* of application. Earth-pull, or gravity, is a special kind of distributed force which is applied throughout the body acted on instead of to a portion of its surface. It will be seen that for some of the purposes of Statics most forces, including gravity, may be regarded as concentrated forces, and that it is convenient to so regard them.

5. Line of Action and Sense. — The *line of action* of a concentrated force (or a force so considered) is a line, indefinite in length, parallel to the direction of the force and containing its point of application. In other words, it is the line along which the force is exerted, or, as we shall say for brevity, the line along which the force acts.

A force may act along its line of action in one of two ways, — to the right or left, up or down, etc. We say that the *sense* of a force is toward the right or left, upward or downward, accordingly. For many purposes a force is sufficiently described by its magnitude, line of action, and sense, since, as will be presently explained, the position of the point of application in the line of action is of no consequence in the problems of Statics.

6. The Occurrence of Forces. — Forces are caused or brought into play in many different ways and occur under an infinite variety of circumstances, but they may be broadly classified according to the way in which they are exerted, as: (i) Forces exerted through contact (such as a push or pull exerted on one solid body by another solid body, or the pressure exerted against a house by the wind, or against a dam by the impounded water), and (ii) Forces exerted without contact (such as the gravitational or magnetic attraction of one body for another). Most forces which, in engineering, have to be dealt with by the principles of statics, are forces exerted on some solid body, either by another solid

body through contact, by a body of gas or liquid through contact, or by the earth through gravitation. In any case it must be remembered that a force is always exerted *on* something *by* something, and one should avoid speaking of a supposed force that cannot be thus accounted for.

Often the direction of the force exerted on one body by another through contact is limited by the *nature of the contact.* Thus if the bodies are smooth, and merely bear one against the other without actual attachment, the force either exerts on the other can only be normal to the surfaces of contact; if the bodies are rough, the force can be inclined more or less according to the degree of roughness;[1] if the bodies are actually connected, as by nails or glue, then the force can have any direction.

The force exerted by a gas or liquid on a solid body through contact is a distributed pressure against the surface of contact. If the body in question is stationary the direction of this pressure is normal to the surface when the gas or liquid is at rest (as in the case of water pressure against a dam), and is often assumed to be so when the gas or liquid is in motion (as in the case of wind pressure against a sloping roof).

The force exerted by the earth through gravitation is, as has been noted, distributed throughout the body acted on, but for many purposes it is correct and convenient to regard this force as a concentrated force acting at a point called the *center of gravity* of the body. The exact significance of the term "center of gravity," and the way in which, for any given body, the position of this point is determined, are explained in Art. 84–87.

7. Action and Reaction. — When one body exerts a force upon another body then the latter also exerts one on the former, and these two forces are equal in magnitude, opposite in sense, and (if concentrated or so regarded) have a common line of action. This fact may be expressed by saying that *action and reaction are equal and opposite,* by action being meant either of the two forces and by reaction the other. This is really a brief statement of Newton's Third Law of Motion (Art. 168); it is given here because it is essential in Statics.

8. The Effects of Forces. — In the introduction to this book Mechanics was defined as the science that treats of the effects of forces, — in particular the effects of forces on solid bodies. It was explained that in general forces have two effects on a body, namely (i) to cause it to move (if at rest) or to move differently (if in motion), and (ii) to deform it. It was also ex-

[1] The components of the force along and perpendicular to the surface of contact are called *friction* and *normal pressure* respectively. It is the friction that tends to prevent the sliding of one body on another, and experience teaches that friction is usually large in the case of rough bodies and small in the case of smooth bodies. If the surfaces in contact were *perfectly* smooth the friction would be nil, and the forces exerted would consist simply of the normal pressures. Of course no actual surfaces are perfectly smooth, but some may, for practical purposes, be so regarded, and for brevity these will be called smooth and those whose resistance to sliding is to be taken into account will be called rough. (See Chap. V for full discussion of friction.)

plained that the study of the relations between forces and the deformations produced by them was the province of Mechanics of Materials; that the study of the relations between forces and motion was the province of Kinetics, and that the study of the circumstances under which a number of forces acting simultaneously on a body mutually counteract each other and result in equilibrium was the province of Statics. In Statics, therefore, we may distinguish as an effect of a force the tendency to preserve or destroy equilibrium. And since supporting or equilibrating forces are produced by the application of a force to a body not free to move, we may also say that one of the effects of a force is to produce, or bring into action, other forces. We often distinguish between the applied, or active, forces and the supporting, or passive, forces they produce, calling the former *loads*, the latter *reactions*. Thus if a beam is placed in a horizontal position with a support under each end and a downward push is applied near the middle, this downward push is termed the load, and the upward forces exerted by the supports are termed the reactions.

In Statics it is really the equilibrating and force-producing effects of a force in which we are interested. It is understood that the body under consideration does not move; we assume that it is rigid,[1] and our problem is to ascertain something as to the forces that hold this rigid body still under the given circumstances — to determine how large they are, or in what direction they act, or where they are applied.

9. The Principle of Transmissibility. — So far as the tendency to maintain or destroy equilibrium is concerned, the position, in the line of action, of the point of application of a concentrated force acting on a rigid body is immaterial. This fact is known as the principle of transmissibility. It may readily be verified by experiment, as, for example, by means of the apparatus represented in Fig. 1. This apparatus consists of a rigid body suspended from two spring balances. The springs are elongated on account of the weight of the body, and if a force, as F, be applied at A, the springs will suffer additional elongations which (in a way) are a measure of the effect of the applied force. If the point of application of F be changed to B or C, the spring readings will not change; hence the effect of F will not have changed.

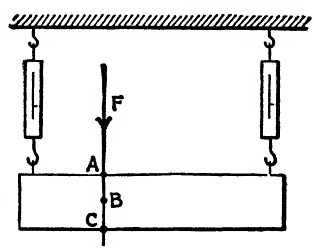

FIG. 1

10. Force Systems. — Any number of forces collectively considered is called a system, or set, of forces. The forces of a set are called *coplanar*

[1] We assume, that is, that the bodies under discussion will not perceptibly bend, stretch, shorten or break as a result of the forces that act on them. We do take into account the fact that certain changes of configuration are *possible*, though they do not occur under the given circumstances; thus a rope or chain is considered as flexible (though inextensible) and joints, hinges, rollers, etc., are considered free to function.

if their lines of action are in the same plane, and *noncoplanar* if they are not in the same plane; they are called *concurrent* if their lines of action intersect at a point, and *nonconcurrent* if they do not so intersect; they are called *parallel* if their lines of action are parallel, and *nonparallel* if their lines of action are not parallel. A given force system is described, in accordance with the foregoing definitions, as concurrent, noncoplanar parallel, etc., according as the forces of the set are concurrent, noncoplanar parallel, etc.

To facilitate systematic discussion, we classify force systems as follows:

Coplanar	concurrent	colinear........	1
		nonparallel.....	2
	nonconcurrent	parallel.........	3
		nonparallel.....	4
Noncoplanar	concurrent..................		5
	nonconcurrent	parallel........	6
		nonparallel.....	7

11. Couples. — A special kind of force system which is of great importance is one that consists of two equal and parallel forces that are opposite in sense. Such a force system is called a *couple*. We here define the term only; properties of couples and principles relating to them are discussed in Art. 32 and 39.

12. The Graphical Representation of Forces. — A force is a vector quantity,[1] and its magnitude and direction can be represented by a vector whose length is proportional to the magnitude of the force and whose direction and sense correspond to those of the force. But a force has also *position* (defined by its line of action, or, if its direction is known, by its place of application), and in order that a force may be fully represented, this too must be indicated. This is done by showing the line of action of the force on a scale drawing of the body on which the force acts.

It is thus seen that the complete graphical representation of a force or a force system may best be accomplished by means of two separate diagrams. One will be made up of the vector or vectors; this is called the *vector diagram*. The other will consist of a scale drawing of the body on which the force or forces act, with the line or lines of action shown thereon; this

[1] A *vector quantity* is any quantity which has both magnitude and direction. Such a quantity can be represented by a *vector*, which is a segment of a straight line of definite length, direction, and sense, the sense being indicated by an arrow head. A vector quantity is to be distinguished from a *scalar* quantity, the latter having magnitude only. Displacement (change of position) is a vector quantity, because a point is displaced in a certain direction as well as through a certain distance; velocity is a vector quantity, because a point moves in a certain direction as well as with a certain speed; force is a vector quantity, because a push or pull is exerted in a certain direction as well as with a certain intensity. On the other hand, volume, area, duration of time, amount of money, are quantities not necessarily associated with direction; each is sufficiently described by its magnitude alone, and these are examples of scalar quantities.

is called a *space diagram.* On the vector diagram one inch represents a certain number of pounds, according to the scale employed in laying off the vectors. On the space diagram one inch represents a certain number of *feet,* according to the scale employed in drawing the body. We could draw the vectors on the space diagram so as to make them indicate the lines of action, and even the points of application, of the corresponding forces, but this is usually not expedient, for, as will be presently seen, the vectors are generally used for a particular purpose that would not be served by such a construction.

In any graphical representation of forces it is convenient to employ a definite system of notation as follows: Each force is designated by two lower case letters placed on opposite sides of its line of action, and the same capital letters placed at opposite ends of the vector that represents it. The force is referred to in written statement by the two capital letters used, the order of these letters indicating the *sense* of the vector, which points from the first to the second.

As an example of the method of graphical representation described above, consider the forces acting on the upper end of the boom of a derrick (Fig. 2). There are two forces; namely, a downward force at pin 1 (equal say to 2 tons) and a force toward the left and upward at pin 2 (equal say to 1.2 tons). The lines marked *ab* and *cd* are respectively the lines of action of the forces. The vectors AB and CD, drawn wherever convenient, parallel respectively to *ab* and *cd*, and equal respectively to 2 tons and 1.2 tons (according to some adopted scale), represent the magnitudes and directions of the forces. The sketch of the boom end and the lines *ab* and *cd* constitute the space diagram; the vectors AB and CD constitute the vector diagram. Though not done in this illustration and not really necessary, it is generally convenient to indicate the sense of a force by an arrow-head placed somewhere on the line of action.

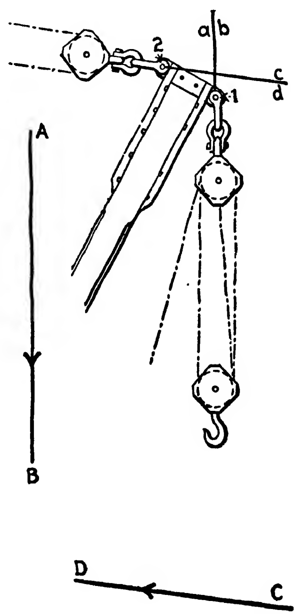

FIG. 2

For greater convenience in discussion, and especially in the statement of a problem, we often designate a force by some single letter, saying "the force F" or "the forces P_1, P_2," etc. The double letter notation described above is used more particularly in the actual graphical solution of problems. For simplicity, in the discussion of principles and general methods we shall represent, in the space diagram, the body on which the forces under consideration act by a subdivided square or rectangle, which

should be thought of as a board or other such *material* thing, and not a mere geometrical figure.

13. Algebraic and Graphical Methods of Analysis. — There are two distinct methods that may be employed in the solution of problems and the discussion of principles in Statics, namely, the algebraic method and the graphical method. When the algebraic method is used, only such sketches are made as are necessary to a clear presentation of the problem and results; quantities are determined by computation. When the graphical method is used the bodies and forces involved are represented, as explained in the preceding article, by drawings made carefully to scale and quantities are determined by measurement, on the drawings, of lengths and angles. Occasions sometimes arise when the two methods may be combined to advantage.

While the algebraic method is better adapted to some cases and the graphical to others, the consideration of both methods usually contributes to the thorough understanding of a problem, and in this book the application of each will (except where the graphical method is impracticable) be explained and illustrated.

CHAPTER II

COMPOSITION AND RESOLUTION OF FORCES

§ 1. Definitions and General Principles

14. Equivalence of Force Systems. — A set of forces which, acting alone on a rigid body at rest, would not cause motion, is said to be in *equilibrium.* Any number of the forces of such a set may be regarded as holding in equilibrium, or balancing, the remainder, — that is, the given force system may be regarded as composed of two systems, A and B, which balance each other. Any other system which would balance A is said to be equivalent to B; any other system that would balance B is said to be equivalent to A. Two force systems are, then, *equivalent* when they have the same *equilibrating* effect.

15. The Resultant of a Force System. — The *resultant* of a force system is the simplest equivalent system. By simplest is meant, containing the fewest forces. It will be seen that the resultant of a force system is either a force, a couple, or two noncoplanar forces, depending upon the nature of the given system. The *anti-resultant* of a force system is the reversed resultant. Obviously the anti-resultant would balance the resultant, hence it would balance the system. Therefore this balancing force is also called the *equilibrant* of the system. If the resultant of a force system is zero, or *nil,* the forces of the system must mutually balance. The following illustration may serve to make clear the physical significance of resultant and equilibrant. Figure 3 represents a ring which is supported by two cords and to which is attached, by means of a third cord, a weight. The ring is acted on by the pulls, F_1 and F_2, of the two inclined cords, and the pull W of the vertical cord. Now the effect of F_1 and F_2 is to balance W; any single force that would balance W would therefore be equivalent to F_1 and F_2 and would constitute their resultant. A force, to balance W, would obviously have to be colinear with, equal, and opposite thereto. Let R be such a force — that is, one equal and opposite to W; R is then the *resultant* of F_1 and F_2. R reversed is the *equilibrant* of F_1 and F_2, which would obviously be identical with W and so, like W, would balance F_1 and F_2. If all three of the forces F_1, F_2 and

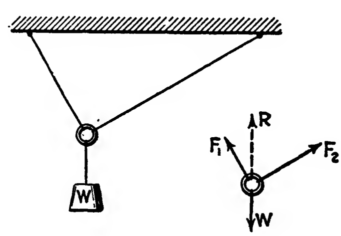

FIG. 3

W are considered, their resultant is the sum of R (the resultant of F_1 and F_2) and W; this sum is zero, as was to be expected, the given forces being in equilibrium.

The process of determining the resultant of a given force system is called *composition*, and the forces are said to be *compounded*. The methods by which composition is effected are discussed in Art. 27–41.

16. The Components of a Force. — If a set of forces is equivalent to a given force, each force of the set may be regarded as a *component* of the given force. The process of determining, for a given force, an equivalent set of components is called *resolution*, and the given force is said to be *resolved* into components.

It will be seen that there are an infinite number of force systems which are equivalent to any given force, that any such system may have any number of forces, and that therefore there are, for a given problem of resolution, an infinite number of different solutions unless restrictions are imposed as to the number, direction, magnitude, etc., of the components. Such restrictions are usually indicated by the conditions of the problem. Particularly important cases are the resolution of a force into two (coplanar) or three (noncoplanar) components that are mutually perpendicular. Such are called *rectangular components.*

The methods by which a force is resolved into components are discussed in Art. 24, 25 and 35.

17. Moment or Torque of a Force. — One can distinguish two somewhat different ways in which the equilibrium of a body may be disturbed: (i) It may be moved, as a whole, up or down, to the right or to the left; or (ii) It may be turned, or rotated, about some fixed line, or axis, without moving away from its general position. Thus a grind-stone may be lifted bodily, or it may be simply turned about the axis of its bearings.

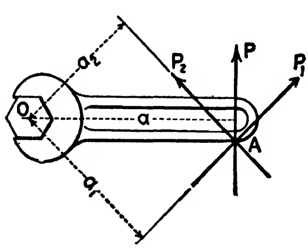

FIG. 4

Obviously the tendency of a force to cause displacement of the first sort is measured wholly by its magnitude. The tendency of a force to turn a body about a given line depends not only upon the magnitude of the force but also upon the position of its line of action with reference to the given line. For example, — common experience teaches that to unscrew a tight nut with a given wrench, one pulls or pushes harder and harder, to the limit of one's strength perhaps; failing to "start" the nut, one gets a larger wrench for its greater "leverage," and to secure the greatest leverage with a given wrench one pushes as indicated by P and not P_1 or P_2 (Fig. 4).

Experiences of the sort just described suggest, and experiments with appropriate laboratory apparatus prove, that the tendency of a force to

turn a body about a line or axis *perpendicular to the force* is directly proportional to (i) the magnitude of the force and (ii) the perpendicular distance between the axis and the line of action of the force; and hence to the product, magnitude of force times perpendicular distance. This product for any particular force and line is called the *moment of the force* about or with respect to that line; and the perpendicular distance is called the *arm* of the force with respect to that line. In Fig. 4 a, a_1 and a_2 respectively are the arms of P, P_1 and P_2 with respect to the axis of the bolt indicated by O.

A force need not be perpendicular to the axis of turning, as in Fig. 4 say, in order to have a turning tendency about that axis. Thus obviously a force P applied to the wrench at A and inclined to the paper tends to turn the nut about the axis of the screw, *unless* the line of action of this force intersects or is parallel to the axis. For this general case, the tendency to turn is measured by the " moment " as defined in Art. 38. Without knowing its precise definition, the student is here expected to accept the term for the general case; to him, it should mean a measure of the tendency of a force to turn a body about a specified line.

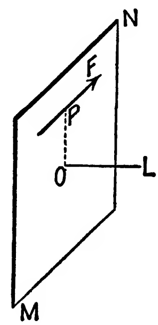

FIG. 5

The moment or torque of a force with respect to a *point* is the product of the force and the perpendicular distance between the line of action of the force and the point. The perpendicular distance is called the *arm* of the force and the point is called a *center* or *origin* of moments. It is apparent that the moment of a force about a point is the same as its moment about a line passing through the point and perpendicular to the plane containing the point and the line of action of the force. In Fig. 5, F is a force in the plane MN and O is an origin of moments in the same plane. OP is the arm of F, and the moment of F with respect to O is equal to $F \times OP$. Obviously this is also the moment of F about the line OL, perpendicular to MN and passing through O.

18. Unit and Sign of Moment. — In accordance with our definition of moment, the unit moment is the moment of a unit force whose arm equals a unit length. There are no one-word names for any of these units of moment; the units are called " foot-pound," " inch-ton," etc., according as the units of length and force used are the foot and the pound, the inch and the ton, etc.

In a discussion involving the moments of several forces about a particular line or origin, it is generally necessary to distinguish between the moments in respect to the direction or sense of the turning which, individually, they tend to produce. Thus we speak of " clockwise moments " and " counterclockwise moments." Whether any particular moment is of one sort or

another depends on the "point of view." Thus the moment of P (Fig. 4) about O is counter-clockwise as the reader views it but if he will turn to page 11 and then look through the sheet at the figure, the moment of P about O is clockwise. In this book, rotation about an axis perpendicular to the printed page is supposed to be viewed from the reading side of such page. Rotation about any other axis is to be viewed as explained in Art. 38. As a convenient means of indicating sense, moments will, in the discussions that follow, be given sign; clockwise moments will be considered negative and counter-clockwise moments will be considered positive.

19. Moment or Torque of a System of Forces. — The moment or torque about any line of a *system of forces* means the algebraic sum of the moments of the forces about that line. Moment of a system of coplanar forces about a point means the algebraic sum of the moments of the forces about that point. We shall represent the moment of a system of forces about a line or point by ΣM_x, ΣM_o, etc., the subscript x, o, etc. indicating the line or point with respect to which moments are taken.

20. The Principle of Moments. — An important principle, which we shall call the principle of moments, may be stated as follows: *If two sets of forces are equivalent, then the moment of one set with respect to any line equals the moment of the other set with respect to the same line.* This follows directly from the definition of *equivalent;* if two force sets are equivalent, they have the same effect in preserving or destroying equilibrium, therefore they have the same turning effect about any given line, therefore they have equal moments with respect to any given line.

It follows from the principle of moments that the moment of any set of forces with respect to a given line equals the moment of the resultant of the set with respect to the same line. Conversely, the moment of a force about any line is equal to the moment of its components with respect to the same line.

For *coplanar* forces the principle of moments may be stated thus: If two sets of coplanar forces are equivalent, the moment of one set with respect to any point in the plane is equal to that of the other set with respect to the same point.

21. Composition of Two Concurrent Forces. — In approaching the problem of composition of forces we consider first the case of two concurrent forces.

Certain facts concerning the resultant of two concurrent forces may almost be said to be apparent. Let F_1 and F_2 (Fig. 6) represent two concurrent forces applied to the square board at points A and B. It would appear to most that a single force, to be equivalent to F_1 and F_2, must pass through their intersection, must lie between them, and must be nearer to the larger force. If we give values to F_1 and F_2, assuming them to be equal respectively to say 10 and 20 pounds, we can, with fair accuracy,

estimate the direction of the resultant R to be about as shown, and its magnitude to be about 25 pounds.

Such conclusions as the above would be reached through the exercise of judgment based on instinct and experience, without recourse to formal analysis, and their validity might therefore fairly be considered as open to question. The conclusion that the resultant passes through the intersection of the component forces is shown to be true by the principle of moments. For the moment of R about any given point must be equal to the moment of its components about that point; the moment of F_1 and F_2 about O is zero; therefore the moment of R about O is also zero; *hence R passes through O.* The *estimated* direction and magnitude of R are at best only approximately correct; the determination of the *exact* direction and magnitude is effected by the method now to be described.

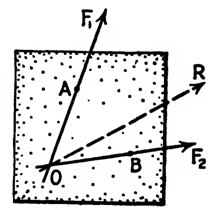

FIG. 6

22. The Parallelogram Law. — If vectors OA and OB represent the magnitude and direction of two concurrent forces, then the vector forming the diagonal OC of the parallelogram constructed on OA and OB as sides represents the magnitude and direction of the resultant of the two forces.

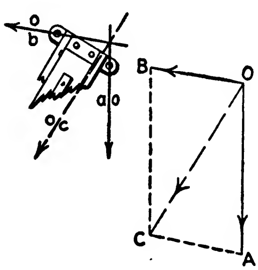

FIG. 7

For example, consider the two forces applied to the cap of the boom, shown in Fig. 2 and discussed in Art. 12. In Fig. 7 are shown the cap and the lines of action of the forces, the latter being marked *oa* and *ob*. Vectors OA and OB, equal respectively to 2 tons and 1.2 tons, are drawn at any convenient place[1] and the parallelogram $OACB$ constructed on these vectors as sides; vector OC then represents the magnitude and direction of the resultant. The magnitude of the resultant is 2.2 tons (found by measuring OC); the direction of the resultant is down and to the left at an angle to the vertical of 33° (found by measuring angle between OC and the vertical).

The *line of action* of the resultant is *oc*, drawn through

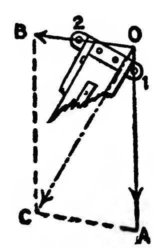

FIG. 8

[1] A practice preferred by some is to construct the parallelogram on the space diagram, laying off the vectors on the corresponding lines of action, as illustrated in Fig. 8. The diagonal OC here represents the resultant completely — direction, magnitude and line of action. At first thought this method may seem superior to that given above; it is really not so good because when several forces are to be compounded, it is likely to lead to confusion as between the space diagram and the vector diagram. It is much better to construct the parallelogram (vector diagram) to one side of the space diagram, so that the two do not overlap, and have no lines in common.

the intersection of *oa* and *ob* parallel to OC. The *sense* of the resultant may be readily apperceived from the space diagram if arrow-heads are placed on the lines of action of the components, as in Fig. 7; it is of course also indicated by the vector OC.

The law which states the relation defined above is called the *Parallelogram Law,* and the construction described is called the *Parallelogram of Forces.* The parallelogram law is to be regarded as a truth based on experiment and observation, and not subject to mathematical demonstration from any more basic fact. It can be " proved " mathematically from the principle of moments, but the principle of moments itself is based on experiment and observation, and so is neither more nor less fundamental than the parallelogram law, which might equally well be used as a basis for a proof of the principle of moments.

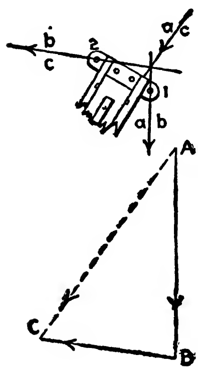

FIG. 9

23. The Triangle of Forces. — Consider the same two forces discussed in the preceding article, but let their lines of action be marked *ab* and *bc* (Fig. 9) and let the vectors representing them, AB and BC, be drawn to form two sides of a triangle ABC. It is obvious that the third side AC of the triangle has the same length and direction as the diagonal OC of the parallelogram of Fig. 7, and so represents the magnitude and direction of the resultant of the given force. The line of action of the resultant is represented by *ac,* drawn through the intersection of *ab* and *bc,* parallel to AC.

This construction, known as the *Triangle of Forces,* has advantages in practical application over the parallelogram construction previously described, in that it involves the drawing of fewer lines and is more readily extended to apply to cases in which more than two forces are involved. It should be noted that when the triangle method is employed the vectors cannot be drawn on the lines of action of the forces they represent.

The triangle of forces suggests an easy *algebraic* method of determining the resultant of two concurrent forces. For the magnitude and direction of the two given forces being known, we have two sides and the included angle of the triangle of forces, and can readily determine the length and the direction of the third side (the resultant vector) by trigonometry. Thus if, in Fig. 9, $AB = 2$ tons, $BC = 1.2$ tons and the angle $ABC = 83°$, then

$$AC^2 = 1.2^2 + 2^2 - 2\ (1.2)\ (2) \cos 83°, \text{ whence } AC = 2.2 \text{ tons.}$$

And also

$$\frac{\sin BAC}{\sin 83°} = \frac{1.2}{2.2}, \text{ whence } BAC = 33°.$$

If the forces to be compounded are at right angles to each other the relations are simplified. Thus if in the above example the angle $ABC = 90°$, then

$$AC^2 = 1.2^2 + 2^2, \text{ whence } AC = 2.33 \text{ tons; and } BAC = \text{arc tan } \frac{1.2}{2.0} = 31°.$$

24. Resolution of a Force into Two Concurrent Components. — A force can be resolved into two concurrent components by applying the parallelogram or triangle law inversely. Thus, let it be required to resolve the force F (Fig. 10) into two components. We draw AB anywhere equal (by some scale) and parallel to F; join any point C with A and B, and draw lines through *any point in ab* parallel to AC and BC; then AC and CB represent the magnitudes and directions, and ac and cb the lines of action of two forces equivalent to F, that is, components of F. For the resultant of these two component forces is F, as shown by the triangle law applied directly. It is particularly important to note that while the given force can be thus resolved into two concurrent components at any point along its line of action, these components *must* intersect *on* the line of action of the given force, that is, *the force and its two components must be concurrent.*

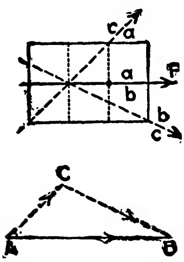

FIG. 10

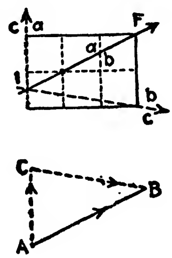

FIG. 11

Since, in the above example, C was taken at random, it is plain that a given force can be resolved into many different pairs of components. If conditions be imposed on the components, the resolution is more or less definite. Thus, let it be required to resolve F (Fig. 11), equal to 350 pounds, into two components, one of which must act along the left-hand edge of the board and the other through the lower right-hand corner. Since the three forces must be concurrent, the second component must act through point 1; so we make AB equal and parallel to F and draw from A and B lines parallel to the two components; then AC and CB represent the values (200 and 320 pounds respectively) and the directions of the components.

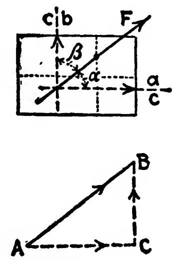

FIG. 12

25. Rectangular Components. — Mention was made in Art. 16 of the important case of resolution in which the components are at right angles to each other, each being called a *rectangular component* or *resolved part* of the force. Rectangular components can generally be computed more easily than found by geometrical construction. Let F (Fig. 12) be the given force to be resolved into horizontal and vertical components, the angles between

F and the components being α and β respectively. From ABC, a sketch of the triangle of resolution, not necessarily a scale drawing, it is plain that the desired components equal $F \cos \alpha$ and $F \cos \beta$ respectively. And always

$$\left.\begin{array}{l}\text{the rectangular com-}\\ \text{ponent of a force}\\ \text{along any line}\end{array}\right\} = \left\{\begin{array}{l}\text{the magnitude}\\ \text{of the force}\end{array}\right\} \times \left\{\begin{array}{l}\text{the cosine of the}\\ \text{acute angle between}\\ \text{the force and that line.}\end{array}\right.$$

The components of a force along two rectangular coördinate axes x and y are called the x and y components of the force respectively; they will be denoted by F_x and F_y. It is convenient to indicate the *sense* of such components by *sign*, an arbitrary rule being adopted and adhered to throughout any given discussion. We shall here call x components positive when they act toward the right and negative when they act toward the left, and y components positive when they act up and negative when they act down.

26. Component of a System of Forces. — Sometimes, for convenience, we will call the algebraic sum of the components, along any line, of the forces of any system, the component, along that line, of the system. Thus, the algebraic sum of the x components of all the forces of a system, we will call the x component of the system and denote by ΣF_x. This is in line with the use of the term "moment of the system" defined in Art. 19.

§ 2. Composition of Various Force Systems.

27. Graphical Composition of Coplanar Concurrent Forces. — We employ the method of *progressive composition*, which may be described as follows: By means of the parallelogram or triangle of forces find the resultant R' of any two of the forces of the given set; then find the resultant R'' of any other given force and R', and so on until the resultant of all is found. Thus, suppose the resultant of the four forces F_1, F_2, F_3 and F_4 (Fig. 13) is required. Taking any two of the forces, as F_1 and F_2, their resultant R' is determined by the force triangle ABC. The resultant R'' of R' and any of the remaining forces, as F_3 is next determined, the force triangle being ACD. Finally, the resultant R''' of R'' and the remaining force F_4 is found, the force triangle being ADE. This resultant R''' is the resultant of the system. In a problem such as that just discussed, one may say that R' *replaces* F_1 and F_2, that R'' replaces R' and F_3, and so forth until the entire system is replaced by its resultant.

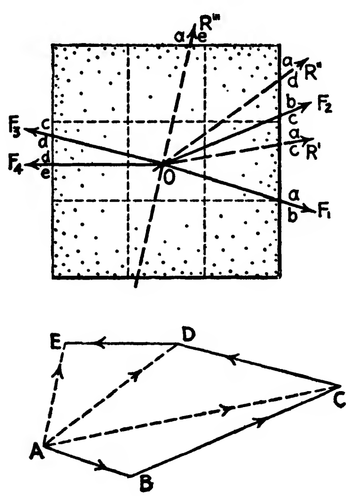

FIG. 13

It is apparent that there is no need to actually determine the successive resultants R', R'', etc.; the lines ac, AC, etc. (Fig. 13) are not essential for determining R''', and are here drawn and referred to only for explanatory purposes. One has only to draw, in sequence and continuously, the vectors for the forces to be compounded; a vector extending from the beginning to the end of the figure so constructed gives the magnitude and direction of the resultant.

The diagram formed by thus drawing in sequence and continuously the vectors for the forces of a system is called the *force polygon* for that system. It is important to note that the vectors must be *confluent*, that is, they must all point the same way around the polygon. In general, the force polygon will not be a closed figure. For any given set of forces as many different force polygons can be drawn as there are orders of taking the forces; they will differ in form, but all will give the same magnitude and direction for the resultant. In Fig. 13, $ABCDE$ is one force polygon for the forces there shown. In Fig. 14 are represented three other force polygons for

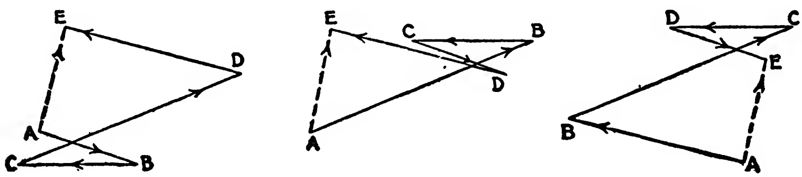

FIG. 14

the same system, the forces being taken in different order. In each case the vector AE is the same and gives the magnitude and direction of the resultant. The bare construction for determining the resultant of a set of concurrent forces can now be stated thus: Draw a polygon for the forces; join the beginning and the end of the polygon, and draw a line through the point of concurrence of the given forces parallel to the joining line; the joining line, with arrow-head pointing from the beginning to end of the force polygon, represents the magnitude and direction of the resultant, and the other line represents its line of action.

28. Algebraic Composition of Coplanar Concurrent Forces. — We proceed as follows: Choose a pair of rectangular axes of resolution, which let us call x and y axes, with origin at the point of concurrence of the forces to be compounded; then resolve each force into its x and y components at the origin, and imagine it replaced by them; the resulting system consists of forces in the x and the y axes; next find the resultant of the forces acting in the x axis, and the resultant of those acting in the y axis; finally, get the resultant of these two rectangular resultants; this is the resultant sought.

EXAMPLE. Six forces act upon a 4 ft. board as shown in Fig. 15. It is required to determine their resultant.

Solution: Rectangular axes are assumed, with origin at the point of concurrence. The angle which each force makes with the horizontal is computed from the dimensions of the figure. Each force is then resolved at the origin into x and y components, and the sum of the x components and the sum of the y components are determined. These computations are outlined in the schedule below.

F	α	$\cos\alpha$	$\sin\alpha$	F_x	F_y
8	0	1.	0.	+8.	0.
4	45°	0.707	0.707	+2.83	+2.83
6	63° 26′	0.447	0.894	−2.68	+5.36
12	36° 52′	0.800	0.606	−9.60	−7.27
7	90°	0.	1.	0.00	−7.00
5	14° 2′	0.970	0.242	+4.85	−1.21
				+3.40	−7.29

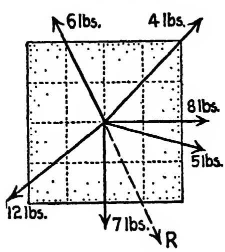

FIG. 15

The resultant of all the x components = +3.40 lbs., and acts to the right (because its sign is plus). The resultant of all the y components = −7.29 lbs., and acts down (because its sign is minus). The resultant of these resultants (and hence of the system) passes through the point of concurrence. It acts to the right and down. The angle it makes with the horizontal is

$$\alpha = \tan^{-1}(7.29 \div 3.40) = 65°$$

Its magnitude is

$$R = (3.40^2 + 7.29^2)^{\frac{1}{2}} = 8.05 \text{ lbs.}$$

It is represented on the figure by R.

29. Graphical Composition of Coplanar Nonconcurrent Forces. — *First method:* When the forces to be compounded are not parallel or nearly so, we can solve by the method of progressive composition, much as in the case of concurrent forces. There being no point of general concurrence, however, the position of the line of action of the resultant is not known in advance. In order that its position may be determined the lines of action of the successive resultants R', R'', etc. must be drawn, on the space diagram. Each such resultant acts through the point of intersection of the forces which it replaces, and its line of action is produced to intersect that of the force with which it is to be compounded. The resultant of the set acts through the intersection of the last force to be compounded and the resultant of all the other forces. This method is illustrated in Ex. 1 below.

EXAMPLE 1. Figure 16 represents a section of a retaining wall, of which wall a portion 1 ft. long is considered. The forces that act on this portion are: Its own weight (16,000 lbs.); the earth pressure on the back (6000 lbs.); that on the top of the base (9000 lbs.), and that on the bottom of the base. It is required to determine the resultant of the first three forces, whose lines of action are known to be as shown.

Solution: The resultant of the 6000 and the 16,000 lb.-forces is determined by means of the force triangle ABC drawn to a certain convenient scale; AB represents

the 6000 lb.-force, BC represents the 16,000 lb.-force, and AC gives the magnitude and direction of their resultant R'. The line of action of R' passes through point 1, and is parallel to AC; it is drawn and produced to intersect the line of action of the 9000 lb.-force. The resultant of R' and the 9000 lb.-force is next determined by means of the force triangle ACD; AC represents R'. CD represents the 9000 lb.-force, and AD gives the magnitude and direction of their resultant R, which is the resultant of the three given forces. The line of action of R passes through point 2 and is parallel to AD, as shown.

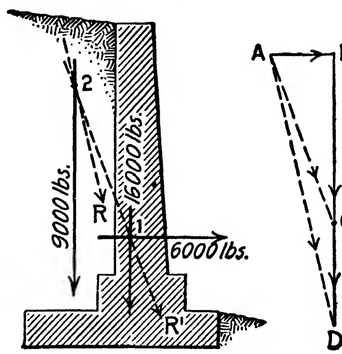

FIG. 16

If the forces to be compounded are parallel or nearly so, the foregoing method fails because there is no accessible intersection of the lines of action of two given forces through which to draw the line of action of the first resultant. This difficulty can be met as follows: Introduce into the given system two equal, opposite, and colinear forces, which will not change the resultant, taking their common line of action somewhat across those of the given forces; then use the first method, compounding first any pair of forces whose intersection is accessible, etc.

Second method: If the system to be compounded contains a considerable number of forces, and these forces are parallel, or so nearly parallel as not to intersect within the limits of the drawing, the following method has certain advantages: First resolve each force into two concurrent components, resolving in such a way that all of the components thus obtained, excepting one of the first force and one of the last force, balance or destroy each other; these two remaining components are, in general, concurrent, and their resultant, which is also the resultant of the given forces, is readily found. This method is illustrated in Ex. 2 below.

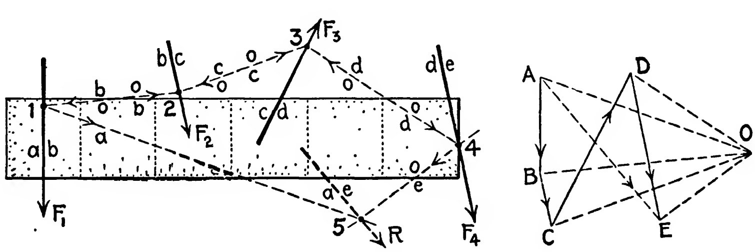

FIG. 17

EXAMPLE 2. Four forces F_1, F_2, F_3 and F_4 act upon a board as shown in Fig. 17. It is required to determine their resultant.

Solution: First a force polygon is drawn for the given forces, these being taken in any convenient order, as $ABCDE$; then any convenient point O is taken as the common vertex of the triangles of resolution. AO and OB represent two components of F_1 in magnitude and direction, BO and OC two components of F_2, etc.; thus this resolution

gives several pairs of equal and opposite components, OB and BO, OC and CO, OD and DO. The components of F_1 are taken to act through point 1, those of F_2 through 2, those of F_3 through 3, etc., the first point, 1, being taken at pleasure on F_1, point 2 where ob intersects F_2, point 3 where oc intersects F_3, etc. Thus the components OB and BO are colinear and they balance; likewise OC and CO, and OD and DO. Only the first and last components AO and OE remain; their resultant is represented by AE in magnitude and direction, and its line of action is ae (parallel to AE through the intersection of ao and oe).

The common vertex of the triangles of resolution O (Fig. 17) is the pole of the force polygon; the lines from the pole to the vertexes of the force polygon, OA, OB, OC, etc., are *rays;* the lines of action of the several forces, oa, ob, oc, etc., are *strings* which, considered collectively, make up the *string* or *funicular polygon* (also called *equilibrium polygon,* especially when the given forces are balanced or in equilibrium). The rays are sometimes referred to by number, OA being the first, OB the second, etc.; likewise the strings.

The bare construction in this second method may be outlined as follows: Draw a force and a string polygon for the forces, then draw a line from the beginning to the end of the force polygon and a parallel line through the intersection of the first and last strings; the first line represents the magnitude and direction of the resultant (sense being from the beginning to the end of the force polygon) and the second line is the line of action of the resultant.

This second method is not so simple in principle as the first, but in the second there is more opportunity for varying the construction to keep the drawing within convenient limits; thus the pole may be shifted, and the starting point of the string polygon may be taken anywhere on any of the given forces. Though many string polygons may be drawn for a given set of forces, all determine the same line of action of the resultant; that is, the intersections of the first and last strings of all string polygons lie on one straight line, the line of action of the resultant.

If a force polygon for a system closes, it may seem at first thought, that the resultant vanishes, or is zero. In general, this conclusion would be wrong. Indeed, the resultant is a couple in general as can be shown readily by means of a simple illustration, say the forces F_1, F_2, F_3 and F_4 (Figs. 18 and 19) which were chosen so that their force polygon $ABCDE$ closes.

Figure 18 shows the construction for finding the resultant of the system by the first method. The resultant R'' of the first three forces is seen to be parallel, equal and opposite to F_4 but not colinear with F_4. Thus the system reduces to a couple (R'' and F_4), and no further composition is possible. Had R'' been found to be colinear with F_4, these two equal, opposite and colinear forces would have balanced, and the resultant of the system would have been shown to really vanish.

Figure 19 shows the construction for the resultant by the second method.

The first and last of the eight components into which the four given forces have been resolved are AO and OE (equal) acting in the lines *ao* and *oe* (parallel). And since the two forces are opposite they constitute a couple, the resultant of the given system. Had *ao* and *oe* been found to be co-

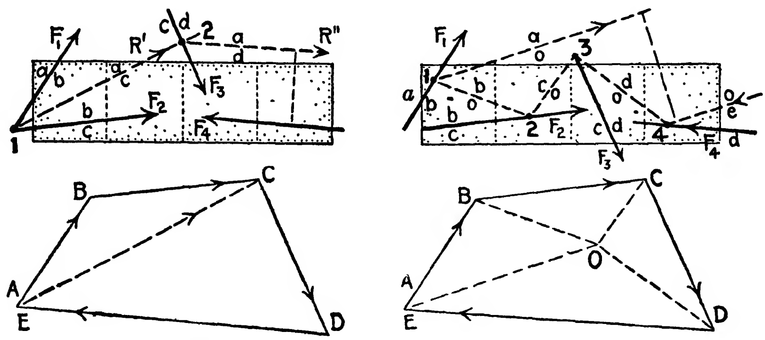

FIG. 18 FIG. 19

linear, or, as we say, had the string polygon been found to close, then the equal, opposite and colinear forces AO and OE would have balanced and the resultant would have been shown to really vanish.

The moment of the resultant couple, according to Fig. 18, is the product of R'' (or F_4) and the perpendicular distance between the lines marked R'' and F_4; according to Fig. 19, it is the product of the force AO (or OE) and the perpendicular distance between the lines *ao* and *oe*. These products are equal. And so any one of an indefinite number of equivalent couples might be found to be the resultant of the given system; the couple arrived at would depend upon the way in which the forces were compounded, but the moment of the couple would not, since it would, in any case, equal the moment of the system.

30. Algebraic Composition of Coplanar Parallel Forces. — If the forces be given sign, those in either direction being called positive and those in the other negative, then the algebraic sum of the forces gives the magnitude and sense of the resultant, the sign of the sum indicating the sense of the resultant. According to the principle of moments (Art. 20) the moment of the resultant about any point equals the algebraic sum of the moments of the forces about that point, and this requirement fixes the position or line of action of the resultant.

If the algebraic sum of a given set of parallel forces equals zero, then it may appear to the student that their resultant vanishes or is zero; this does not follow, but the resultant actually is in general a couple. For the resultant of all but one of the given forces is a single force equal, opposite, and parallel to the omitted one; but these two are not in general colinear, and so they constitute a couple, the resultant of the system. The couple arrived at depends on which one of the given forces is omitted, but the

moment of the couple does not, for that couple is the resultant of the set, and the moment equals the moment of the given set of forces, a definite quantity.

Two parallel forces may of course be compounded by the general method just explained, but the following special results are worth noting, since they facilitate the solution of this very common problem. We distinguish two cases: (i) the two forces are alike in sense; (ii) they are opposite. In (i) the resultant equals the sum of the forces and agrees with them in sense; in (ii) the resultant equals the difference between the two forces and agrees with the larger in sense. In order that the moment of the resultant R may equal the sum of the moments of the forces, P and Q (Fig. 20), then, in case (i), R must lie between the forces, and in case (ii) outside of them and adjacent to the larger force (assumed to be P in the figure). Furthermore, if the distances from R to P and Q be called p and q respectively, and the distance between P and Q be called a, then in either case, $Rp = Qa$ and $Rq = Pa$, or

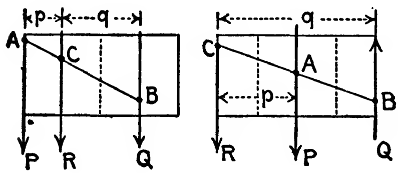

FIG. 20

$$p = Qa/R \text{ and } q = Pa/R$$

either of which definitely fixes the position of R. Also for either case, $Pp = Qq$ or $P/Q = q/p$; hence

$$P/Q = BC/AC,$$

that is, the line of action of the resultant of two parallel forces divides any secant intersecting their lines of action into two segments which are inversely proportional to the two forces.

EXAMPLE. Four parallel forces act on a 10 ft. board as shown in Fig. 21. It is required to determine their resultant.

Solution: First the algebraic sum of the forces is computed. Calling upward forces positive, this sum is

$$\Sigma F = +20 - 40 - 50 + 30 = -40.$$

Hence the resultant equals 40 lbs. and acts down. The moment of the system is next computed for some chosen origin.

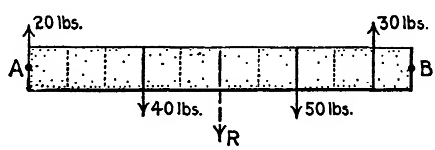

FIG. 21

Taking A as origin and calling counter-clockwise moments positive, this moment is

$$\Sigma M_A = 0 - 120 - 350 + 270 = -200.$$

Hence also the moment of the resultant with respect to A is 200 ft-lbs. and is clockwise. In order to have a clockwise moment a downward force must act to the right of the origin; in order to have a moment of 200 ft-lbs. a force of 40 lbs. must have an arm of 5 ft.; therefore the line of action of the resultant is 5 ft. to the right of A. The resultant is represented on the figure by R.

(The position of R could have been determined equally well using any other point as origin of moments. Thus, taking B as origin, the moment of the system is

$$\Sigma M_B = -200 + 280 + 150 - 30 = 200.$$

Therefore the line of action of R is 5 ft. to the left of B.)

31. Algebraic Composition of Coplanar Nonconcurrent Nonparallel Forces. — We reduce the problem to a combination or series of the simpler cases already considered, proceeding as follows: First, select two rectangular axes of resolution (here called x and y), the position of whose origin is immaterial. Second, resolve each force of the given system into its x and y components and imagine it replaced by them, thus replacing the original force system by two sets of parallel forces, one composed of all the x components, the other of all the y components. Third, determine the resultant of each of these two sets of parallel forces. Fourth, compound these two resultants (which are mutually perpendicular and therefore concurrent) and so determine the resultant of the original system.

It is sometimes more convenient to determine the sense and magnitude only of the resultants of the x and y components, and from these, the direction and magnitude only of the final resultant. The line of action of the resultant is then located by solving for its moment arm with respect to any convenient center of moments. To do this, the moment of all the given forces (or of their rectangular components) with respect to the chosen center is computed, and this moment divided by the resultant gives the arm.

If $\Sigma F_x = 0$, then the resultant of the x components may be a couple; if $\Sigma F_y = 0$ the resultant of the y component may also be a couple. If *both* sums equal zero, the given system may thus be equivalent to two couples; and since the resultant of two couples is a couple (Art. 32), the resultant of the given system is a single couple. The moment of this resultant couple is equal to the moment of the given system about any point.

EXAMPLE. Four forces act on a 4 ft. board as shown in Fig. 22; their points of application are indicated by the heavy dots. It is required to determine the resultant of these four forces.

Solution: Each force is resolved into x and y components; this can be done at any point along the line of action. Here it is done at the point of application. These components are represented in the figure and their values are recorded in the schedule below. The moments of the x and y components are computed for a chosen origin A and are also recorded in the schedule on the following page.

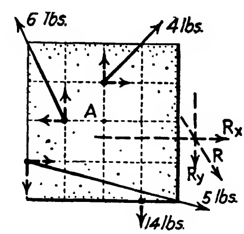

FIG. 22

The sum of the x components is $+5.00$; the moment of the x components about A is $+2.02$ ft-lbs.; the resultant R_x of the x components is therefore a force of 5.00 lbs. acting to the right 0.40 ft. below A.

The sum of the y components is -7.02; the moment of the y components about A is

-16.94 ft-lbs.; the resultant R_y of the y components is therefore a force of 7.02 lbs. acting down 2.41 ft. to the right of A.

Force	α	F_x	F_y	Mom. of F_x	Mom. of F_y
6	63° 26′	−2.68	+5.36	0	−5.36
4	45°	+2.83	+2.83	−2.83	0
5	14° 2′	+4.85	−1.21	+4.85	+2.42
14	90°	0	−14.00	0	−14.00
		+5.00	−7.02	+2.02	−16.94

The resultant of these two resultants (which is the resultant of the original system) acts through their point of concurrence. It is directed to the right and down. The angle it makes with the horizontal is

$$\alpha = \tan^{-1}(7.02 \div 5.00) = 54° 30'.$$

Its magnitude is

$$R = (5.00^2 + 7.02^2)^{\frac{1}{2}} = 8.62 \text{ lbs.}$$

It is represented on the sketch by R.

(Each of the given forces can be resolved into its x and y components at any point along its line of action; the position of R_x and R_y will vary accordingly, but the position of R will not. Also, any origin of moments can be used in locating R_x and R_y without affecting the results. It is possible, however, to so select the points at which the forces are resolved, and the origin about which moments are taken, as to lighten the labor of solution considerably. Thus if the origin of moments is taken in the above example at the intersection of the 4 and 6 lb. forces, and these forces are resolved there, none of these four components need be considered when moments are computed. And the 5 lb. force can be resolved at some point along its line of action so selected as to make either its x or y component pass through the origin of moments.)

The position of R might also have been determined as follows: The moment of the system about $A = 2.02 - 16.94 = -14.92$ ft-lbs. The arm of the resultant is therefore $14.92 \div 8.62 = 1.73$ ft. Since R acts down and to the right and has a negative moment, its line of action must lie above and to the right of A, at a perpendicular distance therefrom of 1.73 ft.

32. Couples. — In Art. 11 a couple was defined as a force system composed of two equal and parallel forces opposite in sense. Such a system cannot be compounded — that is, cannot be reduced to any simpler system. It may, therefore, itself constitute the resultant of a set of coplanar *non-concurrent* forces, as stated in Art. 29, 30 and 31. It has been seen that this is the case when, in a graphical solution, the force polygon closes but the string polygon does not, or when, in an algebraic solution, the sum of the x and y components is zero, but the moment of the set (for any origin that may be chosen) is not zero.

Statement and proof of five important principles relating to couples follow.

(1) *The moment of a couple is the same for all origins in its plane, and is equal to the product of the common magnitude of the forces that compose the couple and the perpendicular distance between their lines of action.* **Proof:**

Let F be the common magnitude of the forces F_1 and F_2 (Fig. 23) that compose a couple acting on a body not shown, and let a be the perpendicular distance between their lines of action (called the *arm of the couple*). Consider three points O_1, O_2, O_3, which represent all possible positions of the moment center. The moments of the couple with respect to these three origins are respectively:

$$-F_2b_1 + F_1(b_1 + a) = +F_1a; \quad +F_2b_2 + F_1(a - b_2) = +F_1a;$$
$$+F_2b_3 - F_1(b_3 - a) = +F_1a.$$

It is seen that these moments are equal and of the same sign, and since O_1, O_2 and O_3 represent all possible origins, the proposition is proved.

It follows from the above that the moment of a couple with respect to any given point in its plane is independent of the position or orientation of the couple in the plane.

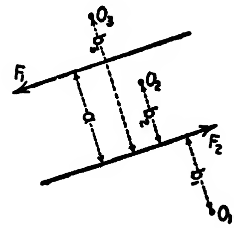

FIG. 23

It is important to remember that to completely describe the moment of a couple, both the magnitude and sign must be specified. To perceive whether the moment of a couple is clockwise or counter-clockwise, it is helpful to imagine the couple applied to a plate or disc pivoted on an axle midway between the forces of the couple; the direction of the rotation which the couple tends to produce is then evident, and the sign of its moment is determined accordingly.

(2) *Two coplanar couples whose moments (including sign) are equal are equivalent.* Proof: Let one couple be composed of forces P_1 and P_2 and the other of forces Q_1 and Q_2, the arms of these couples being respectively p and q. Then $Pp = Qq$. We shall show that the P couple would balance the reversed Q couple by showing that their resultant would be nil; it will follow that the given couples are equivalent.

The P couple and the reversed Q couple constitute a coplanar, nonconcurrent system of four forces which may or may not be parallel. The resultant of such a system (Art. 31) may be a force or a couple. If the resultant is a force it is given by $R = (\Sigma F_x^2 + \Sigma F_y^2)^{\frac{1}{2}}$. But P_1 and P_2 are equal, parallel and opposite; therefore their x components are equal, opposite and add up to zero, and their y components are equal, opposite and add up to zero. The same is true of Q_1 and Q_2. Therefore $\Sigma F_x = 0, \Sigma F_y = 0$ and $R = 0$. The resultant cannot therefore be a force. If the resultant is a couple, its moment is equal to the moment-sum, with respect to any origin, of the force system. But the force system here considered is made up of two couples whose moments, being equal and opposite, add up to zero, and so the resultant cannot be a couple. Therefore the resultant is nil, the P couple balances the reversed Q couple, and the given couples are equivalent.

It follows from (1) and (2) that, so long as the sense and the product force-times-arm are kept the same, a couple can be shifted or turned about in its plane, the forces moved farther apart or closer together, or made larger or smaller, at will, without changing the effect of the couple.

(3) *The resultant of a coplanar force and couple is a single force; it is equal to and has the same direction as the given force and its moment about any point on the line of action of the given force equals the moment of the couple.* Proof: Let the couple, acting on a body not shown, consist of the forces P_1 and P_2 (Fig. 24) with the arm p, and let F be the force. Replace the couple P_1P_2 by an equivalent couple, the forces of which, F_1 and F_2, are equal to F and the moment of which, Ff, must of course equal Pp; then imagine this couple placed so that F_1 is colinear with F. Now F_1 and F, being colinear, equal and opposite, would balance and so vanish; F_2 would remain as the resultant of the system and would be equal to F, parallel to F, and at a distance f therefrom. And its moment about any point in the line of action of F would be F_2f, equal to Pp, the moment of the original couple.

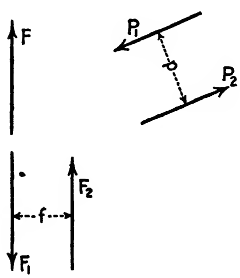

FIG. 24

(4) *A given force can be resolved into a coplanar force and couple; the component force is equal and parallel to the given force and the couple has a moment equal to that of the given force about any point in the line of action of the component force.* Proof: Let F (Fig. 25) represent the given force, which acts on a body not shown. Introduce two equal, opposite and colinear forces F_1 and F_2, making them equal to, parallel to, and coplanar with, F. Since F_1 and F_2 balance, the system composed of the three forces F, F_1 and F_2 is equivalent to F alone. Now F and F_2 constitute a couple; this couple and the force F_1 are equivalent to F, and so the given force F has been resolved into an equal and parallel force F_1 and a couple FF_2 whose moment Ff is equal to the moment of the given force about any point in the line of action of the component force. The couple FF_2 can of course be replaced by any coplanar couple of equal moment.

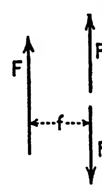

FIG. 25

(5) *The resultant of any number of coplanar couples is a couple.* The given couples constitute a system of coplanar nonconcurrent forces, the force polygon for which closes. The resultant of such a system, as shown in Art. 29, is a couple.

33. Graphical Composition of Noncoplanar Concurrent Forces. — As in the case of coplanar concurrent forces, solution may be effected by the method of progressive composition. Thus, one might determine the resultant R' of any two of the forces, drawing the force triangle in the plane

of these two; then the resultant of R' and some other force, drawing in the plane of these two; and so on until the resultant of all has been determined. Or, one might draw the nonplanar force polygon for the system in plan and elevation and thus determine the resultant.

Neither of these graphical methods is convenient. Indeed, graphical methods for noncoplanar systems of forces are not practical, because the forces are " in space "; algebraic methods are to be preferred. However, we next take up in detail the (graphical) composition of three noncoplanar concurrent forces as a preliminary to algebraic composition.

34. Parallelopiped of Forces. — If three noncoplanar forces acting on a rigid body be represented by OA, OB and OC, then their resultant is represented by the diagonal OD of the parallelopiped $OABC$ (Fig. 26). $OABC$ is called a Parallelopiped of Forces. The relation stated can be established by progressive composition, thus, — The resultant of OA and OB is OC', and the resultant of OC' and OC is OD.

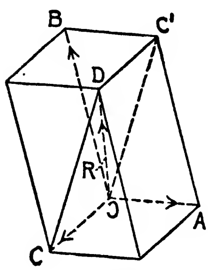

FIG. 26

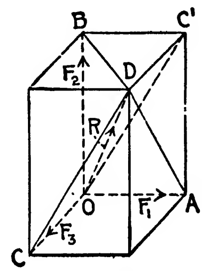

FIG. 27

If the forces to be compounded are mutually perpendicular, the parallelopiped is a " right " one, and it suggests a simple calculation for magnitude and direction angles of the resultant. Thus, let F_1, F_2 and F_3 (Fig. 27) be the three forces, R their resultant, and θ_1, θ_2 and θ_3 the angles between R and the forces respectively; then it is quite apparent from the figure that

$$R^2 = F^2{}_1 + F^2{}_2 + F^2{}_3 \quad \text{and}$$
$$\cos\theta_1 = F_1/R, \qquad \cos\theta_2 = F_2/R, \qquad \cos\theta_3 = F_3/R.$$

35. Resolution of a Force into Three Rectangular Components. — By applying the parallelopiped of forces inversely, one can resolve a given force into three noncoplanar concurrent forces. Obviously an infinite number of different parallelopipeds might be constructed on a given diagonal, hence there is an infinite number of sets of three components unless restrictions are imposed as to direction, magnitude, etc. of the components.

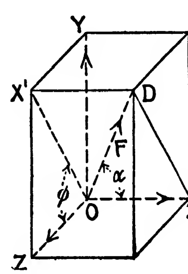

FIG. 28

The practical case is resolution into three rectangular components. In that case algebraic solution is simple; it is based on the parallelopiped. Let F, represented by the vector OD (Fig. 28), be the force to be resolved, α, β and γ the angles between F and a given set of coördinate axes, X, Y and Z through O; also let F_x, F_y and F_z denote the desired components respectively. Since F_x, F_y and F_z are

represented by OX, OY and OZ which are projections of OD on the rectangular axes, it follows that

$$F_x = F \cos \alpha, \qquad F_y = F \cos \beta, \qquad \text{and } F_z = F \cos \gamma.$$

Sometimes the direction of the force F to be resolved is given by means of two angles, one being the angle between F and one of the coördinate axes, and the other being the angle which the projection of F on the plane of the other two axes makes with either of these axes, for example α and ϕ (Fig. 28). When this is the case, F may best be resolved in this way: First resolve F into two components, $F \cos \alpha$ (along the x axis) and $F \sin \alpha$ (in the plane of the y and z axes); then resolve $F \sin \alpha$ along the y and z axes. The final results are

$$F_x = F \cos \alpha, \qquad F_y = F \sin \alpha \sin \phi, \qquad \text{and } F_z = F \sin \alpha \cos \phi.$$

36. Algebraic Composition of Noncoplanar Concurrent Forces. — It has been shown that, by algebraic methods, the resultant of three mutually perpendicular concurrent forces is easily found, and that any given force is easily resolved into three concurrent and mutually perpendicular components. This suggests the following algebraic method by which any number of noncoplanar concurrent forces may be compounded: First select three rectangular axes of resolution (here called x, y and z), with origin at the point of concurrence of the forces to be compounded; next resolve each force into its x, y and z components, and imagine it replaced by them, thus arriving at a set consisting of forces acting in the axes; then find the resultant of the forces in the x, in the y, and in the z axis; finally, compound these three resultants, thus finding the resultant sought.

EXAMPLE. Four forces act on a 4 ft. cube as shown in Fig. 29. It is required to determine their resultant.

Solution: Rectangular axes are assumed as indicated on the figure. Each force is resolved at the origin into x, y and z components. For the 40 and 18 lb.-forces the components are obviously as given in the schedule below. Since the 15 lb.-force is perpendicular to the x axis, its x component is zero; since the angle it makes with the z axis $= \tan^{-1} \frac{3}{4} = 36° 50'$, its y and z components are equal respectively to $15 \sin 36° 50' = 9$ lbs. and $15 \cos 36° 50' = 12$ lbs. Both components are seen by inspection of the figure to be negative. The components of the 10 lb.-force are determined as follows: Since $Ya = 5$ ft., and $YO = 4$ ft., the angle the 10 lb.-force makes with the y axis is $\tan^{-1} \frac{5}{4} = 51° 20'$; therefore the y component of the force equals $10 \cos 51° 20' = 6.25$ lbs., and the other rectangular component (in the xz plane) equals $10 \sin 51° 20' = 7.81$ lbs. The angle which this latter component, acting along Ob, makes with the z axis equals $\tan^{-1} \frac{3}{4} = 36° 50'$; hence the x and z components of the 10 lb.-force are respectively $7.81 \sin 36° 50' = 4.69$ lbs., and $7.81 \cos 36° 50' = 6.25$ lbs. All three components of the 10 lb.-force are seen by inspection to be negative. The values of the rectangular components of the forces and of their algebraic sums are given in the schedule below.

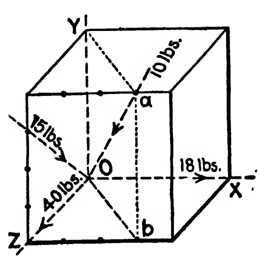

FIG. 29

F	F_x	F_y	F_z
18	18.00	0.00	0.00
40	0.00	0.00	40.00
15	0.00	−9.00	−12.00
10	−4.69	−6.25	−6.25
	+13.31	−15.25	+21.75

The resultant of the x components = +13.31 lbs.; the resultant of the y components = −15.25 lbs.; the resultant of the z components = +21.75 lbs. The resultant of these resultants (and hence of the system) acts through the point of concurrence O. It is directed to the right, down and forward. Its magnitude is

$$R = (13.31^2 + 15.25^2 + 21.75^2)^{\frac{1}{2}} = 29.7 \text{ lbs}$$

The angles it makes with the x, y and z axes are respectively

$$\alpha = \cos^{-1}(13.31 \div 29.7) = 63°$$
$$\beta = \cos^{-1}(15.25 \div 29.7) = 59°$$
$$\gamma = \cos^{-1}(21.75 \div 29.7) = 43°$$

37. Composition of Noncoplanar Parallel Forces. — It is shown in Art. 30 that the resultant of any two parallel forces is parallel to those forces, and that its magnitude and sense are given by the algebraic sum of the forces, the sense being given by the sign of the sum. It follows that the resultant of any number of parallel forces, coplanar or noncoplanar, is parallel to the forces, and that its magnitude and sense are given by the algebraic sum of the forces (all forces of the same sense having one sign, and those of the opposite sense having the opposite sign). The line of action of the resultant may be fixed by means of the arms of the resultant with respect to two rectangular axes, each perpendicular to the forces. Such arms can be computed readily from the principle that the moment of the resultant about any axis equals the algebraic sum of the moments of the forces about the same axis.

When the algebraic sum of the forces is zero, it may seem at first thought that the resultant is zero; in general this conclusion would be wrong, the system actually reducing to a couple. For the resultant of all the forces save one will be parallel, equal, and opposite to that force, and (unless colinear with it, in which case the resultant really is zero) will constitute with it a couple. The magnitude of the forces, the arm, and the plane of the resultant couple will depend on which one of the given forces is last compounded, but all resultant couples that may be determined will be found to lie in parallel planes and to have equal moments, and so are equivalent (see (1) of Art. 39).

EXAMPLE. Five vertical forces act upon a horizontal rectangular platform 4 ft. by 6 ft. as shown in Fig. 30. It is required to determine their resultant.

Solution: The algebraic sum of the forces is computed; this sum is

$$\Sigma F_y = 15 - 40 - 20 + 25 - 30 = -50.$$

Hence the resultant is a force of 50 lbs. and acts down. The moment of the system is computed for each of two chosen axes perpendicular to the forces. For the axes x and z (the back edge and left edge, respectively, of the platform) these moments are:

$$\Sigma M_x = -(15 \times 4) + (40 \times 3) + (20 \times 1) - (25 \times 2) + (30 \times 3) = +120, \text{ and}$$
$$\Sigma M_z = +(15 \times 1) - (40 \times 2) - (20 \times 3) + (25 \times 5) - (30 \times 6) = -180.$$

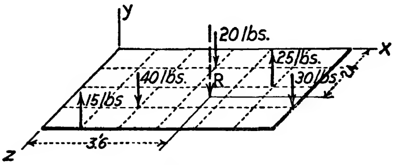

FIG. 30

In order to have a positive moment of 120 ft-lbs. about the x axis, the (downward) resultant must act in front of the x axis and at a distance therefrom of $120 \div 50 = 2.4$ ft. In order to have a negative moment of 180 ft-lbs. about the z axis the resultant must act to the right of the z axis and at a distance therefrom of $180 \div 50 = 3.6$ ft. Therefore the resultant of the system is a downward force of 50 lbs. acting 2.4 ft. in front of the back edge, and 3.6 ft. to the right of the left edge, of the platform. It is represented in the figure by R.

38. Moment of a Force About a Line. — In Art. 17 we discussed this subject but left unfinished the discussion for the case where the force and the line are *not* at right angles. We resume at this place.

As pointed out in that article, a force has no tendency to turn the body on which the force acts, about a given line or axis if the line of action of the force is parallel to or intersects the axis. And so to arrive at a measure of the turning tendency of any force, one resolves the force into two components, one parallel to or intersecting the axis and the other perpendicular to the axis, and then computes the moment of the second component. Because the first component has no tendency to turn the body, this computed moment measures the turning tendency of the original force. Hence, the following extension of the definition in Art. 17 to the general case:—

The moment of a force about any specified line is the moment of its rectangular component perpendicular to the line, the other component being parallel to or intersecting the line.

Generally it is more convenient to resolve the force into two components so that one is parallel to the axis. Thus, let F (Fig. 31) acting on a body not shown be the force, and LL' the line or axis of moments. MN is a plane perpendicular to the line, represented merely to make the figure plain. OB is drawn parallel to LL'; OA is drawn perpendicular to OB, and in the plane of OB and OC; the parallelogram $OABC$ is constructed on the given diagonal OC. Then OA and OB are the desired components, and the moment of F is the product of the component OA and the perpendicular PL (between OA and LL').

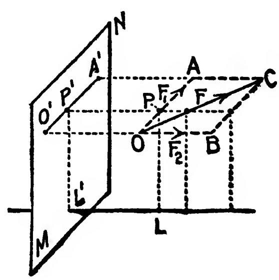

FIG. 31

Obviously the projection of the vector OC on the plane MN is a vector

equal to the component OA, and the perpendicular distance from LL' to the projection $O'A'$ is the same as the arm PL of the component OA. Hence the moment of the original force F is also equal to the product $O'A' \times P'L'$.

39. Couples. — Article 32 relates to couples and forces in one plane; we now extend the discussion to apply to couples in space. Statement and proof of important facts concerning couples in space follow:

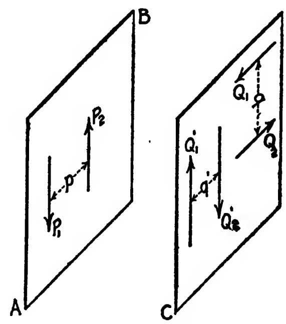

FIG. 32

(1) *Two couples whose planes are parallel and whose moments (including sign) are equal are equivalent.* Proof: Let one couple be composed of the forces P_1 and P_2 in the plane AB (Fig. 32), and the other of the forces Q_1 and Q_2 in the parallel plane CD; the couples act on a body not shown. The arms of the couples are respectively p and q, and $Pp = Qq$. We shall show that the P couple would balance the reversed Q couple by showing that the resultant of the two would be nil; it will follow that the given couples are equivalent.

Imagine the Q couple reversed; then replace the reversed Q couple by the equivalent couple $Q'_1Q'_2$, making Q'_1 and Q'_2 parallel to P_1 and P_2. Q'_1, Q'_2, P_1, and P_2 now constitute a noncoplanar parallel system; the resultant of such a system (Art. 37) may be either a force or a couple. If the resultant is a force it is equal to the algebraic sum of the given forces, and since Q'_1 and Q'_2 are equal and opposite and P_1 and P_2 are equal and opposite, this algebraic sum is zero. Therefore the resultant is not a force. If the resultant is a couple, its moment is equal to the moment of the given forces. But the force system here considered is made up of two couples whose moments, being equal and opposite, add up to zero. Therefore the resultant is not a couple. And so the resultant is nil; the P couple balances the reversed Q couple, and the given couples are equivalent.

(2) *The resultant of any number of couples is a couple.* Proof: We show that the resultant of any two couples is a couple. It follows that the resultant of any number of couples is a couple, for this resultant would be found by getting the resultant of any two of the given couples, compounding this resultant with another of the given couples, and so on, the resultant in every instance being a couple.

Imagine each of the given couples turned so that its forces are parallel to the line on which its plane intersects that of the other couple. The couples now constitute a system of four parallel forces. The algebraic sum of these four forces is zero (since the system consists of two pairs of equal and opposite forces) and therefore, as shown in Art. 37, the resultant (if not zero) is a couple.

If the given couples are in parallel planes, then they can be replaced by equivalent couples all in the same plane, and the resultant of these equivalent couples is a couple (Art. 32).

40. Vector Representation and Composition of Couples. — In discussions involving couples in nonparallel planes it is sometimes convenient to represent the couples by vectors. The vector of a given couple is perpendicular to the plane of the couple (exact position of vector immaterial); its length is equal to the moment of the couple according to some scale understood; and its sense agrees with the sense (rotation) of the couple according to some rule of agreement, as for example the following: Imagine the vector to be a right-handed screw (in a fixed nut) turning with the couple; then the arrow-head on the vector must point in the direction in which the screw advances. For example the couple PP in the plane MN (Fig. 33) is represented by the vector AB (perpendicular to the plane MN), whose length equals Pp according to some scale.

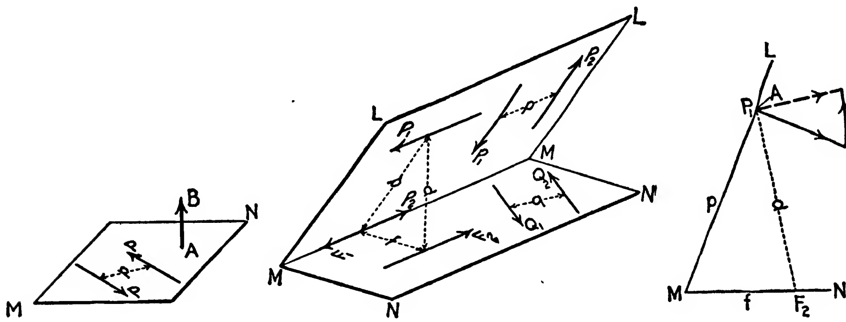

FIG. 33 FIG. 34 FIG. 35

The vector of the resultant of any number of couples equals the sum of the vectors of those couples. We prove this proposition for two couples; it follows that it is true for any number. Let P_1P_2 and Q_1Q_2 (Fig. 34) represent couples in the planes LM and MN respectively. Replace the Q couple by an equivalent couple F_1F_2, making $F = P$, and place the F couple and the P couple so that one force of each lies in the line of intersection of the planes, MM, these colinear forces (here F_1 and P_2) being opposite in sense. Obviously F_1 and P_2 balance, and so F_2 and P_1 constitute a couple which is the resultant of the original two couples. Now let Fig. 35 represent an end view of the planes LM and MN, looking along their line of intersection MM. Let vector AB (drawn from any point on P_1) represent the couple P_1P_2 and the vector BC the couple F_1F_2; the forces of these couples being equal their moments are proportional to their arms p and f respectively, and so the length of AB is proportional to p and the length of BC is proportional to f. The (vector) sum of AB and BC is AC; the triangles P_1MF_2 and ABC are similar, therefore AC is

proportional to d (the arm of the resultant couple F_2P_1) and perpendicular to the plane of F_2P_1, and so represents this resultant couple.

41. Composition of Nonconcurrent Noncoplanar Nonparallel Forces. — A set of noncoplanar, nonconcurrent, nonparallel forces may be compounded, in general, into a force acting through a chosen point and a couple. Proof follows: As explained in Art. 32, each force of the given system may be resolved into and be replaced by a force acting through the chosen point and a couple. Supposing such a replacement made for each given force, then the new system consists of a set of concurrent forces at the given point and a set of couples; but the resultant of the concurrent forces is a single force acting through the chosen point (Art. 36), and the resultant of the couples is a single couple (Art. 39). This force and couple respectively will be denoted by R and C, and it will now be shown how R and C can be determined.

Let F_1, F_2, F_3, etc. (Fig. 36, only F_1 shown), be the forces of the given system acting on a body not shown; O the point through which R is to pass; and OX, OY and OZ any convenient axes of reference. Let P_1 and Q_1, acting at O (Fig. 37), be equal and parallel to F_1; similarly, let P_2 and Q_2 (not shown) act at O, and be equal and parallel to F_2; etc. Then the force P_1 and the couple F_1Q_1 (Fig. 37) are equivalent to F_1 (Fig. 36); the force P_2 and the couple F_2Q_2 are equivalent to F_2; etc. To determine R, note that the axial components of P_1, P_2, P_3, etc. (the concurrent forces), are respectively equal to the axial components of F_1, F_2, F_3, etc. (the given forces); hence if ΣF_x, ΣF_y and ΣF_z denote the alge-

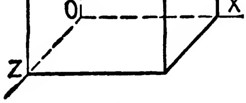

FIG. 36

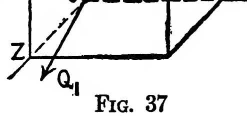

FIG. 37

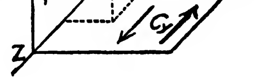

FIG. 38

braic sums of the x, y and z components of the given forces, then $R_x = \Sigma F_x$, $R_y = \Sigma F_y$ and $R_z = \Sigma F_z$; also

$$R^2 = (\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

And if θ_1, θ_2 and θ_3 denote the direction angles of R, then

$$\cos\theta_1 = \Sigma F_x/R, \qquad \cos\theta_2 = \Sigma F_y/R, \qquad \cos\theta_3 = \Sigma F_z/R \quad . \; . \quad (2)$$

To determine C: Imagine it resolved into three components whose planes are respectively perpendicular to the x, y and z axes (Art. 40), and denote the components and their moments by C_x, C_y and C_z (Fig. 38).

Since the system R, C_x, C_y and C_z is equivalent to the given system, their moments about any line are equal; hence $C_x = \Sigma M_x$, $C_y = \Sigma M_y$, and $C_z = \Sigma M_z$, where ΣM_x, ΣM_y and ΣM_z denote the moments of the given system with respect to the x, y and z axes respectively. Also, according to Art. 40,

$$C^2 = (\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2; \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

and if ϕ_1, ϕ_2 and ϕ_3 denote the direction angles of the vector representing C, then

$$\cos\phi_1 = \Sigma M_x/C, \quad \cos\phi_2 = \Sigma M_y/C, \quad \cos\phi_3 = \Sigma M_z/C. \; . \; . \; . \; (4)$$

In general, R and C may be compounded into two noncoplanar forces. For, as explained in Art. 39, C may be shifted about without change of effect if only the direction of its plane be unchanged; assume such shift until one of the forces of C intersects R; then that force and R may be compounded into a single force R'; there remain R' and the second force of C, and obviously R' and that force are not coplanar. These two cannot be compounded; they are the simplest set equivalent to the given system, and therefore constitute the resultant of the given system. If the plane of C happens to be parallel to R, then C and R can be compounded into a single force, and the resultant of the given system is a single force. For shifting C about until C and R become coplanar, then they may be compounded readily into a single force (Art. 32).

In general, the system of forces has a torque about every line through O. There is one line which is of prime importance, the line about which the torque is greatest. The torque of the forces about that line is called *the* torque or the resultant torque of the system (for the chosen point O). Since R has no moment about a line through O, the torque of the system about any such line equals the torque of C about that line. But the torque of C is greatest about a line perpendicular to the plane of C; this is the important line mentioned. The direction of this line is given by equations (4), and the resultant torque of the system by equations (3). The system of forces has no torque about a line through O parallel to the plane of C (perpendicular to the line or axis of resultant torque), since R and C have no torque about such line.

CHAPTER III

FORCES IN EQUILIBRIUM

§ 1. Principles of Equilibrium

42. Preliminary. — In Art. 14 it was explained that a set of forces which, acting alone on a rigid body at rest, would not cause motion, is said to be in equilibrium. Sometimes it is important to know whether or not a given set of forces is in equilibrium; again, it is at times important to determine something concerning some of the forces of a set which is known to be in equilibrium. To answer either type of question requires a knowledge of the *conditions* which a force system must satisfy in order that it shall be in equilibrium, and these conditions will now be considered.

43. The General Conditions of Equilibrium. — If a system of forces is in equilibrium the resultant of the system is *nil;* this is the general condition of equilibrium for any kind of force system. This general condition implies subordinate conditions; thus, for any system whatever:

(A) *The algebraic sum of the (rectangular) components of all the forces along any line equals zero,* and

(B) *The algebraic sum of the moments of all the forces about any line (for coplanar forces, about any point) equals zero.*

By means of (A) and (B) one can write many equations for any system in equilibrium. Thus, for a coplanar concurrent system, (A) gives $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_u = 0$, etc., where x, y, u, etc., are axes of resolution; and (B) gives $\Sigma M_a = 0$, $\Sigma M_b = 0$, $\Sigma M_c = 0$, etc., where a, b, c, etc., are origins of moments in the plane of the forces. Not all of such equilibrium equations are independent, however; that is, certain ones follow from the others. Thus, if $\Sigma F_x = 0$ for any coplanar concurrent system, then ΣF_y does not necessarily equal zero, but if also $\Sigma F_y = 0$, then the resultant equals zero, and it *follows* that $\Sigma F_u = 0$. That is, $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are two independent equations, but any third similar equation (as $\Sigma F_u = 0$) is not independent of them. The independent equations or conditions of equilibrium for any particular kind of force system are such as are necessary and sufficient to insure a vanishing resultant. We will now deduce these independent conditions of equilibrium for the various classes or kinds of force systems, expressing them first on the algebraic basis and secondly (for coplanar forces) on the graphical basis.

44. Colinear Forces. — There is one condition of equilibrium. It can be stated in several forms; namely,

$$(1)\ \Sigma F = 0 \qquad \text{or} \qquad (2)\ \Sigma M_a = 0.$$

Form (1) states that the algebraic sum of the forces **equals zero**; (2) that the algebraic sum of the moments of all the forces about any point (not on their common line of action) equals zero.

On the graphical basis, the condition of equilibrium is that the force polygon for the forces (degenerated into a straight line in this case) is a closed one. For if $\Sigma F = 0$, or $\Sigma M = 0$, or the force polygon closes, then there is no resultant.

45. Coplanar Concurrent Forces. — There are two independent algebraic conditions of equilibrium. They can be expressed in three forms; namely,

$$(1)\ \Sigma F_x = \Sigma F_y = 0, \quad (2)\ \Sigma F_x = \Sigma M_a = 0, \quad \text{or} \quad (3)\ \Sigma M_a = \Sigma M_b = 0.$$

Form (1) states that the algebraic sums of the components of the forces along two lines x and y (in the plane of the forces) equal zero; (2) that the algebraic sum of the components of the forces along any line (as x), and the algebraic sum of the moments of all the forces about any point, each equals zero (the point a to be in the plane of the forces, and the line joining a and O, their point of concurrence, to be inclined to the x axis); and (3) that the algebraic sums of the moments of all the forces about two points (not colinear with the point of concurrence of the forces) equal zero. For in any case the resultant is zero, as will be seen from this: (1) According to Art. 28, the resultant of the system, if there is one, is a single force R, given by $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$; and hence if $\Sigma F_x = 0$ and $\Sigma F_y = 0$, R must equal zero. (2) If $\Sigma F_x = 0$, then the resultant, if there is one, must be perpendicular to the x axis; and if $\Sigma M_a = 0$, then the moment of R about a equals zero, which requires that $R = 0$. (3) The resultant, if there is one, must pass through the point of concurrence O of the given forces; if $\Sigma M_a = 0$ then R must pass through a also; if $\Sigma M_b = 0$, then R must equal zero, b not being on Oa.

The graphical condition for equilibrium is that the force polygon for the forces closes. For, if it does close, there is no resultant.

46. Coplanar Nonconcurrent Parallel Forces. — There are two independent algebraic conditions of equilibrium. They can be expressed in two forms; namely,

$$(1)\ \Sigma F = \Sigma M = 0 \qquad \text{or} \qquad (2)\ \Sigma M_a = \Sigma M_b = 0.$$

Form (1) states that the algebraic sum of the forces and the algebraic sum of the moments of the forces about any point (in the plane of the forces) equal zero; (2) that the algebraic sums of the moments of the forces about two points equal zero, the line joining the origins not to be parallel to the forces. For either set of conditions is necessary and sufficient to make the resultant zero, as may be shown thus: In Art. 30 it is shown that the resultant, if there is one, is a single force or a couple. And (1), if $\Sigma F = 0$, then the resultant is not a force, and if $\Sigma M = 0$, then it is not a couple;

and hence there is no resultant. (2) If $\Sigma M_a = 0$, the resultant is not a couple but a force, which passes through a; if also $\Sigma M_b = 0$, then the moment of the resultant force about b must be zero, and that requires that the force equals zero.

There are two graphical conditions of equilibrium, namely, a force and a string polygon for the forces must close. For if a force polygon closes, then the resultant, if there is one, is a couple; if a string polygon closes, then the resultant is not a couple.

47. Coplanar Nonconcurrent Nonparallel Forces. — There are three independent algebraic conditions of equilibrium. They can be stated in three forms; namely,

$$\begin{aligned}
&(1)\ \Sigma F_x = \Sigma F_y = \Sigma M_a = 0;\\
&(2)\ \Sigma F_x = \Sigma M_a = \Sigma M_b = 0;\\
&(3)\ \Sigma M_a = \Sigma M_b = \Sigma M_c = 0.
\end{aligned}$$

Form (1) states that the algebraic sums of the components of all the forces along two lines and the algebraic sum of the moments of the forces about any point equal zero, the lines and points to be in the plane of the forces; (2) that the algebraic sum of the components of the forces along any line x and the algebraic sums of the moments of the forces about two points, a and b, equal zero, the line x and that joining a and b not to be at right angles; and (3) that the algebraic sums of the moments of the forces about three points, a, b and c, equal zero, the points not to be colinear. For any set of these conditions is necessary and just sufficient to make the resultant vanish as may be shown, thus: The resultant, if there is one, is a single force or a single couple (Art. 31). And (1) if $\Sigma F_x = \Sigma F_y = 0$, then the resultant is not a force, and if $\Sigma M = 0$, it is not a couple; and hence there is no resultant. (2) If $\Sigma F_x = 0$, the resultant is a force R perpendicular to the x axis or a couple; if $\Sigma M_a = 0$, it is not a couple, but a force passing through a (and perpendicular to the x axis); if also $\Sigma M_b = 0$, then the moment of that force about b must equal zero, and hence the force must equal zero. (3) If $\Sigma M_a = 0$, the resultant, if there is one, is not a couple but a force passing through a; if $\Sigma M_b = 0$, that resultant passes through b; if also $\Sigma M_c = 0$, then the resultant force must equal zero.

There are two graphical conditions, just like those for parallel coplanar nonconcurrent forces; namely, a force and a string polygon must close. For if a force polygon closes, then the resultant, if there is one, is not a force but a couple; if a string polygon closes, then the resultant is not a couple, and so there is no resultant (see Art. 29).

48. Noncoplanar Concurrent Forces. — There are three independent algebraic conditions of equilibrium. The convenient form is

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0;$$

that is, the algebraic sums of the components of all the forces along three rectangular axes, x, y and z, equal zero. For as shown in Art. 36, the

resultant, if there is one, equals $\sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$, and so if the conditions stated are fulfilled then the resultant equals zero.

49. Noncoplanar Parallel Forces. — There are three independent algebraic conditions of equilibrium. There are two convenient forms; namely,

$$(1)\ \Sigma F = \Sigma M_1 = \Sigma M_2 = 0, \qquad \text{and} \qquad (2)\ \Sigma M_1 = \Sigma M_2 = \Sigma M_3 = 0.$$

Form (1) states that the algebraic sum of the forces and the algebraic sums of the moments of the forces about two lines perpendicular to the forces but not parallel to each other equal zero; (2) that the algebraic sums of the moments about three coplanar nonconcurrent nonparallel lines perpendicular to the forces equal zero. For (1) if $\Sigma F = 0$, the resultant is not a force; if $\Sigma M_1 = 0$, the resultant is a couple whose plane is parallel to the first line or axis of moments (and to the forces); and if $\Sigma M_2 = 0$, then the plane of the couple must also be parallel to the second axis; but all these conditions of parallelism cannot be fulfilled unless the two forces of the couple are colinear, in which case the two forces balance, so that there is really no resultant. (2) If $\Sigma M_1 = \Sigma M_2 = 0$, then the resultant must be a force passing through the intersection of lines 1 and 2; if $\Sigma M_3 = 0$, then that force must equal zero, that is, the three conditions make the resultant vanish.

50. Noncoplanar Nonconcurrent Nonparallel Forces. — There are six independent algebraic conditions of equilibrium, namely,

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0;$$

that is, the algebraic sums of the components of all the forces along three lines and the algebraic sums of the moments of the forces about three noncoplanar axes equal zero. (It is generally most convenient to take the three lines and the three axes at right angles to each other.) For the resultant of the system, if there is one, is always reducible to a single force and a single couple (Art. 41); if $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$, the single force equals zero, and if $\Sigma M_x = \Sigma M_y = \Sigma M_z = 0$, then the couple vanishes, and so there is no resultant.

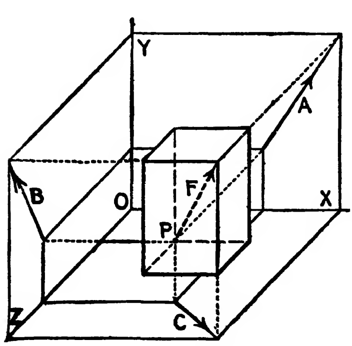

FIG. 39

If every force in the given system (in equilibrium) be represented by a vector, and all these vectors be projected on three rectangular coördinate planes, then the three sets of projections represent three force systems, and each is in equilibrium (proved below). In some cases it may be more convenient to deal with these projected systems. In general, each furnishes three conditions or equations of equilibrium, making nine in all; but

there are duplicates among the nine, and only six are independent. To prove the foregoing, let F (Fig. 39) be one of the forces of the system in equilibrium and P its point of application (on a body not shown). A, B and C are projections of the vector F on the xy, yz and zx planes respectively. Obviously, the x and y components of A equal F_x and F_y respectively; the y and z components of B equal F_y and F_z respectively, and the z and x components of C equal F_z and F_x respectively, as indicated. Since the given system is in equilibrium,

$$\begin{array}{llll} (1) & \Sigma F_x = 0, & (4) & \Sigma M_x = \Sigma(F_z y - F_y z) = 0, \\ (2) & \Sigma F_y = 0, & (5) & \Sigma M_y = \Sigma(F_x z - F_z x) = 0, \quad \text{and} \\ (3) & \Sigma F_z = 0, & (6) & \Sigma M_z = \Sigma(F_y x - F_x y) = 0. \end{array}$$

Now ΣF_x is also the sum of the x components of the A-system; ΣF_y is also the sum of the y components of the A-system; and $\Sigma(F_y x - F_x y)$ is also the sum of the moments of the A forces about O. Hence (1), (2) and (6) are conditions which assert the equilibrium of the A-system. For similar reasons (2), (3) and (4) assert the equilibrium of the B-system and (1), (3), (5) assert the equilibrium of the C-system.

51. Special Conditions of Equilibrium. — Certain special conditions of equilibrium, depending on the number of forces in the system, will be found of great use. They are as follows:

(1) A *single force* cannot be in equilibrium.

(2) If *two forces* are in equilibrium, then obviously they must be colinear, equal and opposite.

(3) If *three forces* are in equilibrium, then they must be coplanar, and concurrent or parallel. Proof: Let the three forces be called F_1, F_2 and F_3; since F_1 and F_2 balance F_3, F_1 and F_2 have a single force resultant R colinear with F_3; since F_1 and F_2 have a resultant colinear with F_3, they lie in a plane with F_3. If F_1 and F_2 are concurrent, then R is concurrent with them and hence F_3 also; if F_1 and F_2 are parallel, then R and hence F_3 is parallel to them.

When the three forces are concurrent, then each is proportional to the sine of either angle between the other two (Lami's theorem); that is,

$$\frac{F_1}{\sin\alpha' = \sin\alpha''} = \frac{F_2}{\sin\beta' = \sin\beta''} = \frac{F_3}{\sin\gamma' = \sin\gamma''},$$

where F_1, F_2 and F_3 are the forces, α' and α'' the (supplementary) angles between F_2 and F_3, β' and β'' those between F_1 and F_3, and γ' and γ'' those between F_1 and F_2. These equations follow at once from application of the sine law to a force triangle for the three forces. When the three forces are parallel, then the two outer ones act in the same direction and the middle one in the opposite direction, and the moments of any two of the forces about a point on the third are equal in magnitude and opposite in sense, or sign.

(4) If *four coplanar forces* are in equilibrium, then the resultant R of any two of the forces balances the other two. Hence, (a) if the first two are concurrent and the second two also, then R passes through the two points of concurrence; (b) if either two are concurrent and the other two parallel, then the resultant R of the first pair acts through the point of concurrence and is parallel to the second pair; (c) if all four forces are parallel, then R is parallel to the forces. Principles (a) and (b) are useful in graphical analysis of four-force systems.

52. Summary. — The *algebraic conditions* of equilibrium explained in detail in the foregoing are brought together here for convenience of reference.

Coplanar Forces.

Colinear, $\Sigma F = 0$; or $\Sigma M = 0$.

Concurrent, $\Sigma F_x = \Sigma F_y = 0$; or $\Sigma F_x = \Sigma M_a = 0$; or $\Sigma M_a = \Sigma M_b = 0$.

Parallel, $\Sigma F = \Sigma M = 0$; or $\Sigma M_a = \Sigma M_b = 0$.

Nonconcurrent nonparallel, $\Sigma F_x = \Sigma F_y = \Sigma M = 0$; or

$\Sigma F_x = \Sigma M_a = \Sigma M_b = 0$; or $\Sigma M_a = \Sigma M_b = \Sigma M_c = 0$.

Noncoplanar Forces.

Concurrent, $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.

Parallel, $\Sigma F = \Sigma M_1 = \Sigma M_2 = 0$; or $\Sigma M_1 = \Sigma M_2 = \Sigma M_3 = 0$.

Nonconcurrent nonparallel, $\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$.

The *graphical conditions* of equilibrium for coplanar systems: for concurrent forces, the force polygon closes; for nonconcurrent forces, the force and the string polygon close. There are graphical conditions of equilibrium for noncoplanar forces, but their usefulness is very limited, and they are therefore not given here.

53. External and Internal Forces. — The word *body* is used in Mechanics in a broad sense to denote any *definite* portion of matter, whether simple and rigid, like a stone or log, or complex, like a bridge or locomotive, or fluid, like the steam in a boiler or the water in a pond. And, in accordance with this definition, we call any *part* of a stone, log, bridge, etc., a body if that part is of special interest and under separate consideration.

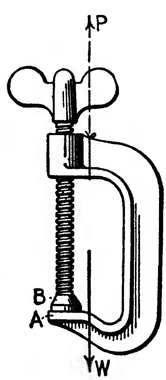

FIG. 40

It is convenient to distinguish between forces, as *external* or *internal* with reference to any particular body: A force is said to be *external* to a body if it is exerted on that body by some other body; and *internal* if the force is exerted on a part of that body by some other part of the same body. For example, consider the screw clamp in Fig. 40, suspended by means of a string from a support above, and imagine that the screw is turned down hard so that the parts A and B are tightly pressed together. With reference to the *entire clamp*, the upward pull P of the string, and the down-

ward pull W of the earth (weight of clamp) are external forces; the upward push Q of A on B, and the equal downward push Q' of B on A are internal forces. With reference to the *screw* considered alone, Q is an external force, and with reference to the *frame* alone, Q' is an external force.

With reference to any given body all the external forces, collectively considered, constitute what is called the external system, and all the internal forces, the internal system. While all forces occur in pairs the forces of which are equal, opposite and colinear ("action and reaction," see Art. 7), there is an important distinction between the internal and the external systems of forces in respect to this duality, namely, — *both forces* of each pair occur in the internal system but only *one force* of each pair in the external. For example, in the screw clamp, both Q and Q' belong to the internal system whereas the reactions corresponding to the forces P and W do not belong to the external system, because they are not exerted on the clamp, but on the string and the earth respectively.

All the forces that act on a body at rest, external and internal, constitute a system in equilibrium (Art. 14). But since the internal system, consisting of pairs of equal, opposite and colinear forces, is balanced, it follows that the external system also is balanced. Hence it may be stated that *the external system of forces acting on any body at rest is in equilibrium.*

54. The Free Body Diagram. — The following articles of this chapter deal with applications of the conditions of equilibrium, usually for the purpose of ascertaining desired information concerning certain forces. The conditions must of course be applied to a system in equilibrium, therefore their use requires the prior establishing or setting up of a system which is in equilibrium and which includes one or more of the forces to be studied. This is best done by first considering a *body* that is known to be at rest and to be acted on by one or more of the forces to be studied, and then identifying the system of external forces that act on this body; this system will be in equilibrium and will include one or more of the forces to be studied. A device or scheme which is of great aid in doing this is the so-called *Free Body Diagram,* now to be described.

The free body diagram is a sketch showing (i) *the body in question by itself entirely isolated from other bodies,* and (ii) *all the external forces exerted on that body.*

The free body diagram must show the body in question and no other body; it must show all the external forces that act on the body in question and no other forces. Those external forces that are known should be fully represented; those that are not known should be represented as fully as present information permits. Care should be taken to guard against showing forces exerted *by,* instead of *on,* the body, and to avoid showing internal forces. It is to be remembered that an external force is one exerted on the body in question by something else, in general

either by the earth through gravitation or by some body through contact. It follows that the number of external forces is in general equal to one plus the number of contacts between the given body and other bodies. (It is here assumed that there is no " force without contact " except gravity.) The logical procedure is to represent the wholly known forces first, indicating the magnitude, line of action, and sense of each; the unknown forces are then represented as fully as possible. Consideration of the nature of the contact at a place where an unknown force is applied often suggests information concerning that force (see Art. 6). If the direction of a force is not known, the force may be shown as acting at an unknown angle with the horizontal (or other reference line), or it may be represented by rectangular components. More detailed directions for constructing a free body diagram are given in connection with the examples below.

EXAMPLE 1. A cylinder rests in a trough formed by two smooth inclined planes. (Fig. 41.) It is required to construct the free body diagram for the cylinder.

Solution: A separate sketch of the cylinder alone is made (Fig. 42). The weight of the cylinder, which acts vertically downward through its center, is represented by W.

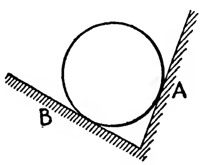

FIG. 41

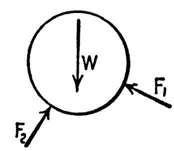

FIG. 42

The places where the cylinder is in contact with other bodies are noted; these are at A and B. At A the plane exerts a force upon the cylinder; since the surfaces in contact are smooth, this force is a push normal to the plane; it is represented by F_1. At B the plane exerts a force upon the cylinder; since the surfaces in contact are smooth, this force also is a push normal to the plane. It is represented by F_2. Figure 42 is the complete free body diagram for the cylinder.

EXAMPLE 2. A block A rests on the rough, inclined top of a second block B, which in turn rests on a smooth, horizontal floor (Fig. 43). It is required to construct the free body diagram for: (*a*) The block A; (*b*) the block B; (*c*) the two blocks considered together as a single body.

Solution: (*a*) A separate sketch is made of block A (Fig. 44). The external forces acting on A are its own weight and the reaction exerted by B. Since these two are the only external forces they must be opposite and colinear, therefore the reaction (represented by N) acts up in the line of action of the weight (represented by W)[1]. Figure 44 is the complete free body diagram for block A.

(*b*) A separate sketch is made of block B (Fig. 45). The external forces acting on B are its own weight, represented by W'; the pressure exerted by block A, equal and opposite to the force N exerted by B on A (action and reaction) and so represented by N;

[1] It is not always possible to represent an unknown force on the free body diagram in its correct position, and in many cases it is immaterial whether it is so represented or not. But the student will find it instructive to determine, so far as can be done by inspection, the approximate position of each force and to represent it accordingly.

and the vertical upward push exerted by the smooth floor, represented by M. This force M must lie somewhere between W' and N, as it is the equilibrant of these two forces (Art. 30). Figure 45 is the complete free body diagram for block B. (It should be noted that while B supports A, the *weight* of A does not act on B; it acts on A. And so

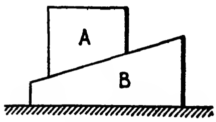

FIG. 43

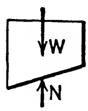

FIG. 44

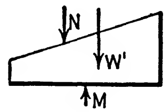

FIG. 45

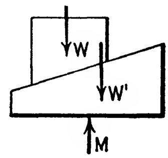

FIG. 46

in all cases, — the *weight of a body acts on the body itself.* But it may *cause* forces to be exerted on other bodies, as here the weight of A causes A to exert the force N on B.)

(*c*) A sketch is made of A and B together (Fig. 46). The external forces acting are the weight of A, represented by W; the weight of B, represented by W'; and the upward reaction of the floor represented by M. Figure 46 is the complete free body diagram for the two blocks considered as a single body. (It should be noted that the force exerted by A on B and that exerted by B on A are not represented, because these are *internal* forces with respect to the body here considered.)

EXAMPLE 3. A beam (Fig. 47) is pinned to the wall at A, held by a cord (weight negligible) at B, and supports a sphere which rests between it and the wall. All surfaces in contact are smooth. It is required to construct the free body diagram for: (*a*) The entire system, comprising beam, sphere and cord, regarded as a single body; (*b*) the group comprising the beam and sphere; (*c*) the sphere; (*d*) the beam.

Solution (*a*) The free body diagram for the entire system is shown in Fig. 48. The external forces are: The weight of the sphere W; the weight of the beam W'; the pull T of the wall at E (known to be along the axis of the cord because the latter is

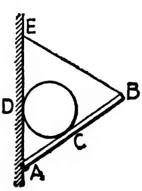

FIG. 47

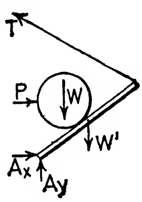

FIG. 48

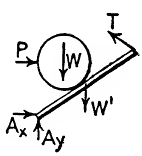

FIG. 49

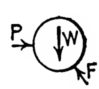

FIG. 50

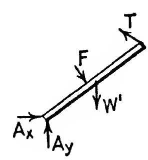

FIG. 51

a two-force member);[1] the push P of the wall at D (known to be a horizontal push because the smooth vertical wall can only exert a normal pressure); the force exerted by the pin at A, the direction of which is unknown and which is therefore represented by its rectangular components A_x and A_y, with senses assumed.

(*b*) The free body diagram for the group comprising the beam and sphere is shown in Fig. 49. The external forces are: The weight of the sphere W; the weight of the beam W'; the pull T of the cord at B; the push P of the wall at D; the components A_x and A_y of the force exerted by the pin at A.

[1] It is often convenient to designate a body, according to the number of external forces that act on it, as a one-force body (or member, or piece), two-force body, three-force body, etc. Thus in the above example the cord is a two-force body, the sphere a three-force body, and the beam a four-force body. This designation of bodies according to the number of external forces that act on them suggests the application of the special conditions of equilibrium given in Art. 51.

(c) The free body diagram for the sphere is shown in Fig. 50. The external forces are: The weight of the sphere W; the push of the wall P; the reaction of the beam F (known to be normal because the beam is smooth).

(d) The free body diagram for the beam is shown in Fig. 51. The external forces are: The weight of the beam W'; the pull of the cord T; the pressure of the sphere F (equal and opposite to force F acting on sphere); the components A_x and A_y of the force exerted by the pin at A.

EXAMPLE 4. The structure shown in Fig. 52 consists of two rigid pieces (weights negligible) pinned to the floor at A and at B and to each other at C. A load is suspended from D as shown. It is required to construct the free body diagram for the entire system, that is, the structure and the load.

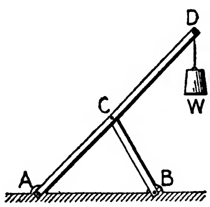

FIG. 52

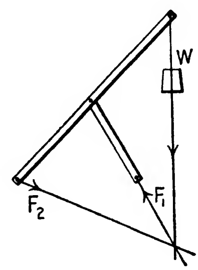

FIG. 53

Solution: A sketch of the structure and attached load is made (Fig. 53). The external forces acting on this body are the weight of the load, represented by W, the force exerted by the pin at A, and the force exerted by the pin at B. Since BC is a two-force member, the forces at its ends are colinear, and so the force at B acts along BC. It is represented by F_1, with sense assumed. The force at A is unknown as to direction (it cannot be assumed to act along the axis of ACD, because this is not a two-force member). It could be represented by its x and y components with senses assumed, as in Ex. 3, but here the external system comprises but three forces which, since they are not parallel (W and F_1 are not parallel), must be concurrent (Art. 51); hence the force at A must act through the intersection of W and F_1. It is represented by F_2, with sense assumed. (Here the senses of F_1 and F_2 are really evident,[1] and this is often true in the case of forces which, so far as the manner of their application is concerned, might act either way along their line of action. Frequently, however, the sense of such a force is not evident, and it will be seen later that an erroneous assumption with respect to the sense is of no consequence. For the sake of the training involved, however, the student is urged to *judge* the sense, rather than to make a mere guess at it.)

§ 2. Applications of the Principles of Equilibrium

55. Statement and Solution of the General Problem. — The common application of the principles of equilibrium is to the solution of a type of problem which may be stated and solved as follows:

Problem. — A given body is at rest while other bodies exert forces on it. Some of these forces are wholly known, some are partially or wholly

[1] That is, they may readily be determined by inspection, without recourse to actual computation. Thus F_1 must act as shown to balance the moment of W about A, and F_2 must act as shown to balance the moment of W about B.

unknown. Some or all of the unknown quantities (magnitude, line of action, or sense) are to be determined.

Solution. — (i) Make a free body diagram for the given body. (ii) Apply appropriate conditions of equilibrium to the system of external forces of the diagram and so determine the unknowns.

By "given body" is really meant a *selected body,* the selection being made so that (i) the external system of forces exerted on that body includes one or more forces to be determined, and (ii) the external system is really solvable for some or all of the desired unknowns.[1] The making of the free body diagram is the crucial step in the solution, — indeed, the importance of this diagram can scarcely be overstated, as it serves to at once define the problem, show the kind of force system involved, present the available data, and indicate the quantities to be determined.

"Appropriate conditions" are such as pertain to the particular kind of system at hand (coplanar concurrent, coplanar parallel, etc.; see Art. 44–52). "Applying" algebraic conditions means writing out or setting up the equations of equilibrium, and solving them simultaneously for the unknowns they contain. "Applying" graphical conditions means drawing the closed force polygon, and if appropriate, the closed string polygon; such construction determines the unknowns.

The application of the general method of solution here described to problems involving the several kinds of force systems and requiring the determination of various unknown quantities is discussed in succeeding articles and illustrated by typical examples.

56. Coplanar Concurrent Forces in Equilibrium. — The conditions of equilibrium for coplanar concurrent forces are stated in Art. 45. To show how these conditions are applied we now distinguish and separately discuss two typical cases.

(1) All the forces except two are wholly known; the lines of action of these two are known and their magnitudes and senses are to be determined.

EXAMPLE 1. Two straight rigid bars *MO* and *NO* (Fig. 54) are pinned to the floor at their lower ends and to each other at their upper ends. A load *W* of 1000 lbs. is suspended from the pin at *O* and a horizontal pull of 800 lbs. is also applied to this pin as shown. It is required to determine the forces exerted on the pin at *O* by the two bars. (It may be assumed that the weights of the pin and bars are negligible and that the axes of the bars and cord all lie in the same vertical plane.)

[1] In the discussion of the conditions of equilibrium (Art. 44–52) it was seen that for any given kind of force system there are a certain number of independent conditions of equilibrium, and therefore an equal number of independent equations that could be written. In any algebraic solution one requires as many independent equations as there are unknown quantities to be solved for. Therefore in any equilibrium problem solution by means of equilibrium equations is impossible whenever the number of unknown quantities exceeds the number of independent conditions of equilibrium. And in such cases solution by graphical methods is also impossible, for in graphical solutions we really make use of the same conditions as in algebraic solutions, although the statement of these conditions takes a different form.

Algebraic solution: The free body diagram for the pin at O is constructed (Fig. 55). The forces exerted on the pin are the horizontal pull of 800 lbs., the downward pull of 1000 lbs., the force exerted by MO, and the force exerted by NO. These last two forces act along MO and NO respectively (because MO and NO are two-force members), but their magnitudes and senses are unknown. Their senses (in this case not evident) are *assumed*,[1] and the forces represented by F_1 and F_2.

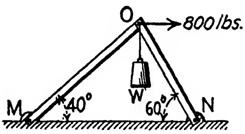

FIG. 54

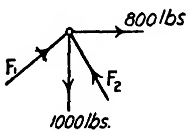

FIG. 55

The force system is seen to be coplanar and concurrent; the appropriate conditions of equilibrium are $\Sigma F_x = \Sigma F_y = 0$, and the application of these conditions yields the following equations:

$$\Sigma F_x = F_1 \cos 40^\circ - F_2 \cos 60^\circ + 800 = 0, \text{ and}$$
$$\Sigma F_y = F_1 \sin 40^\circ + F_2 \sin 60^\circ - 1000 = 0.$$

These equations, solved simultaneously, give

$$F_1 = -196 \text{ lbs.} \quad \text{and} \quad F_2 = 1300 \text{ lbs.}$$

The negative sign shows that the sense of F_1 is opposite to that assumed; this is indicated on the figure by crossing out the assumed arrow-head. The positive sign shows that the sense of F_2 is as assumed.[2]

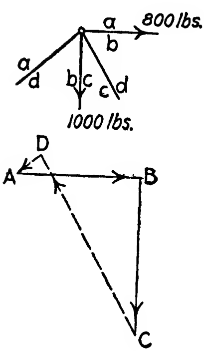

FIG. 56

Graphical solution: The free body diagram is constructed as before, except that the senses of the unknown forces are not assumed (Fig. 56). The lines of action of the forces are lettered in accordance with the usual convention and for convenience this is done in alphabetical order, — *ab*, *bc*, *cd*, etc., — the *known* forces being lettered first. The force polygon is then started, the vectors for the known forces AB and BC being drawn. From A a line is drawn parallel to the line of action of the unknown force DA; from C a line is drawn parallel to the line of action of the unknown force CD; the intersection of these lines closes the force polygon and fixes the point D. The arrows are confluent in the force polygon, therefore CD = 1300 lbs. (scaled) and acts up and to the left, and DA = 196 lbs. (scaled) and acts down and to the left.

(2) All the forces except one are wholly known; the magnitude, sense and line of action of this one are to be determined.

[1] Whenever a force whose sense is unknown is to be entered in a resolution or moment equation, a sense should be assumed for that force and adhered to in the solution of the equation. The correct sense is indicated by the sign of the computed value of that force; a positive sign indicates that the sense assumed is correct and a negative sign that the sense assumed is wrong. Senses found to be wrong are corrected in the figures of the book, by a short line across the assumed arrowhead (Fig. 55).

[2] The student is urged to always record the results of a solution such as the above on the free body diagram, underscoring or otherwise marking these results so as to distinguish them from the given data. When this is done, the free body diagram shows at a glance the values obtained for the unknown quantities. This is not done in the examples in this book, because it is here desired to place emphasis on the *given* data and to avoid any possibility of confusion between the quantities that are originally known and the quantities that are solved for.

EXAMPLE 2. A weight W of 100 lbs. is suspended from a ring; the ring in turn is suspended from a ceiling, and has applied to it a pull of 20 lbs. inclined 30° to the horizontal as shown in Fig. 57. It is required to determine the tension[1] in the cord that

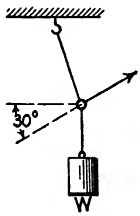

FIG. 57

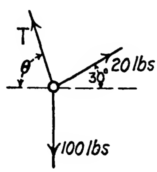

FIG. 58

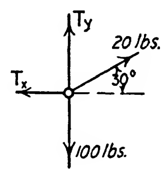

FIG. 59

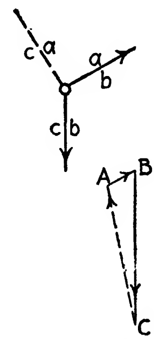

FIG. 60

supports the ring from the ceiling and the position assumed by this cord under the given circumstances.

Algebraic solution: The free body diagram for the ring is constructed (Fig. 58). The external forces that act on the ring are: The pull of the cord by which the weight is suspended, acting down and equal to 100 lbs.; the applied force of 20 lbs. at 30° to the horizontal; the pull exerted by the supporting cord, whose magnitude and direction are unknown and which is represented by T, making an angle θ to the horizontal. The force system is seen to be coplanar and concurrent; appropriate conditions of equilibrium are $\Sigma F_x = \Sigma F_y = 0$; application of these conditions yields the equations

$$\Sigma F_x = 20 \cos 30^\circ - T \cos \theta = 0 \quad \text{and}$$
$$\Sigma F_y = -100 + 20 \sin 30^\circ + T \sin \theta = 0.$$

These equations, solved simultaneously, give

$$T = 91.6 \text{ lbs.} \quad \text{and} \quad \theta = 79^\circ\, 10'.$$

The direction of T is, of course, the same as that of the cord that exerts it.

Alternative algebraic solution: Instead of representing the unknown force as in Fig. 58, it may be considered replaced by its rectangular components T_x and T_y (Fig. 59). The problem is now like (1) above. Application of the conditions $\Sigma F_x = \Sigma F_y = 0$ yields the following equations:

$$\Sigma F_x = 20 \cos 30^\circ - T_x = 0 \quad \text{and}$$
$$\Sigma F_y = -100 + 20 \sin 30^\circ + T_y = 0.$$

These equations give $T_x = 17.3$ and $T_y = 90$. T is the resultant of T_x and T_y, therefore

$$T = (17.3^2 + 90^2)^{\frac{1}{2}} = 91.6 \text{ lbs.},$$

and the angle it makes with the horizontal is

$$\theta = \tan^{-1}(90 \div 17.3) = 79^\circ\, 10'.$$

---

[1] "Tension in a cord" refers to the forces which two parts of a taut cord exert upon each other at any imagined transverse plane of separation. Each of these forces is equal to the pull at either end of the rope. (See Art. 61 for full discussion.)

Graphical solution: The free body diagram is drawn and lettered, the unknown line of action of T being drawn broken (Fig. 60). The force polygon is started, the vectors for the *known* forces AB and BC being drawn. The closing vector CA gives the direction and magnitude of T.

57. Coplanar Parallel Forces in Equilibrium.

— The conditions of equilibrium for coplanar parallel forces are stated in Art. 46. To show how these conditions are applied we now distinguish and separately discuss two typical cases.

(1) All the forces except two are wholly known; the lines of action of these two are known and their magnitudes and senses are to be determined.

EXAMPLE 1. A horizontal beam 16 ft. long is supported at the right end and 6 ft. from the left end, and carries three concentrated loads, one of 2000 lbs. at the left end, one of 1000 lbs. at 4 ft. from the left end, and one of 3000 lbs. at 9 ft. from the left end. The weight of the beam is negligible. It is required to determine the reactions of the supports.

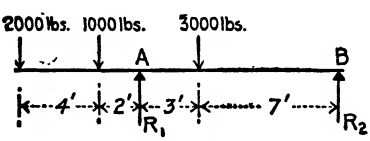

FIG. 61

Algebraic solution: The free body diagram for the beam is shown in Fig. 61. The external forces are the three known loads and the two unknown reactions, all of which are vertical; it is assumed that the reactions act upwards. It is seen that the system is coplanar and parallel; appropriate conditions of equilibrium (Art. 46) are $\Sigma M_a = M_b = 0$; application of these conditions with moment origins A and B taken on the lines of action of R_1 and R_2 respectively (in order to thus eliminate one of the unknown forces from each moment equation), yields the following equations:

$$\Sigma M_A = (2000 \times 6) + (1000 \times 2) - (3000 \times 3) + (R_2 \times 10) = 0$$
$$\Sigma M_B = (2000 \times 16) + (1000 \times 12) + (3000 \times 7) - (R_1 \times 10) = 0.$$

The first equation gives $R_2 = -500$ lbs., and the second gives $R_1 = +6500$ lbs.; the negative sign shows that the sense of R_2 is opposite to that assumed and that it therefore acts down; the positive sign shows that the sense of R_1 is as assumed and that it therefore acts up. As a *check* on the solution the condition $\Sigma F_y = 0$ is applied; thus,

$$\Sigma F_y = -2000 - 1000 - 3000 + 6500 - 500 = 0.$$

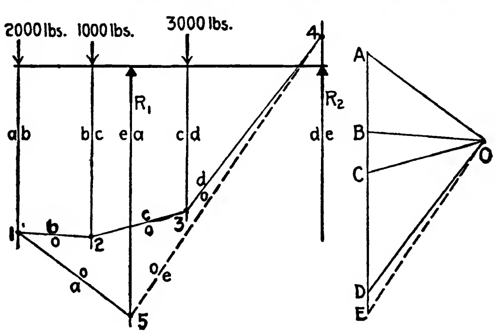

FIG. 62

Graphical solution: The graphical conditions of equilibrium are: The force polygon must close; the string polygon must close. The process of constructing and closing the polygons determines the unknown forces. The free body diagram is constructed and lettered (Fig. 62). The force polygon is drawn as far as possible, the vectors for

the known forces AB, BC and CD being laid off (since some of the vectors of the straight-line force polygon will overlap, the arrow-heads are omitted to avoid possible confusion). The unknown force DE (R_2) and EA (R_1) close the polygon, therefore the problem reduces to that of finding the position of E on the force polygon. A pole O is chosen and the rays AO, BO, CO and DO are drawn. If the direction of the ray EO were known, it could be drawn and point E thus located; we determine the direction of the ray OE by ascertaining the direction of the corresponding string oe in the string polygon. (It should here be recalled that the strings represent the lines of action of the components into which the forces are supposed to be resolved, while the rays are the vectors of these components.)

The string polygon may be started at any point on any of the lines of action of the forces of the system. It is here started at 1, on ab; then strings oa and ob must be drawn through 1 (since AO and OB are components of AB); oc must be drawn from 2 (where ob cuts bc); od from 3 (where oc cuts cd); and oe from 4 (where od cuts de) and through 5 (where oa cuts ea). Thus oe closes the string polygon. The ray OE is then drawn from O parallel to the string oe, thus locating E. DE then represents R_2 and EA represents R_1.

(2) All the forces except one are wholly known; the magnitude, sense and line of action of this one are to be determined.

EXAMPLE 2. A horizontal beam 12 ft. long is supported at the left end and at one other point. It carries two loads, one of 10,000 lbs. at 4 ft. from the left end and one of 8000 lbs. at 9 ft. from the left end. The weight of the beam itself is negligible. The reaction at the left end of the beam is known to act upward and to be equal to 4000 lbs. It is required to determine the position and value of the other reaction.

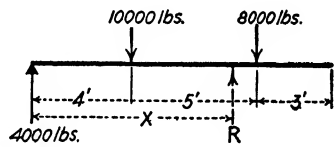

FIG. 63

Algebraic solution: The free body diagram for the beam is shown in Fig. 63. The unknown reaction R is assumed to act upward and the distance from the left end of the beam to its line of action is called x. Appropriate conditions of equilibrium are $\Sigma F = \Sigma M = 0$. Application of the first condition give

$$\Sigma F = +4000 - 10{,}000 - 8000 + R = 0, \text{ whence } R = 14{,}000 \text{ lbs.}$$

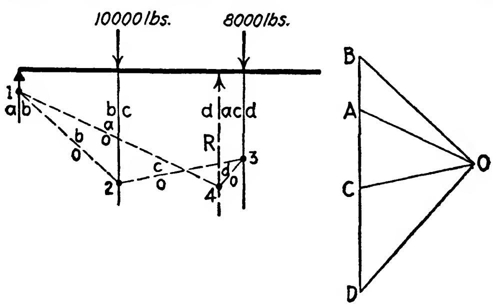

FIG. 64

Application of the second condition, with the origin of moments taken at the left end of the beam, gives

$$\Sigma M = -(10{,}000 \times 4) - (8000 \times 9) + (R \times x) = 0, \text{ whence } x = 8 \text{ ft.}$$

Graphical solution: The free body diagram is drawn and lettered (Fig. 64) the unknown line of action of R being omitted. The force polygon is constructed, the vectors

for the known forces AB, BC and CD being drawn first; the closing vector DA gives the magnitude of R. To determine the position of R, the string polygon is employed, the line of action of R being located by the intersection of the two strings which represent the lines of action of the two components into which R is resolved. A pole O is chosen and rays AO, BO, CO and DO drawn. The string polygon can be started at any point on the lines of action of the known forces; it is here started at 1 on ab. Since all the rays are given, the string polygon can be drawn completely. The strings do and ao are the lines of action of DO and OA, the components of DA (R). Therefore R must act through 4, the intersection of these strings, and so its line of action is determined.

58. Coplanar Nonconcurrent Nonparallel Forces in Equilibrium.— The conditions of equilibrium for coplanar nonconcurrent nonparallel forces are stated in Art. 47. To show how these conditions are applied we now distinguish and separately discuss two typical cases.

(1) All the forces except three are wholly known; only the lines of action of these three are known, and their magnitudes and senses are to be determined.[1]

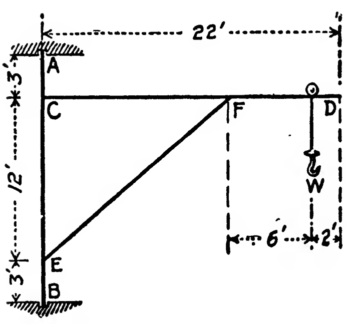

FIG. 65

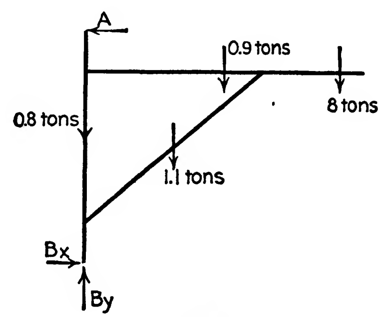

FIG. 66

EXAMPLE 1. Figure 65 represents a crane consisting of a post AB, a boom CD, and a brace EF, which weigh respectively 0.8, 0.9 and 1.1 tons. The post rests in a depression in the floor below and against the side of a hole in the floor above. A load W of 8 tons is suspended from the hook as shown. It is required to determine the reactions of the floors at A and at B.

Algebraic solution: The free-body diagram for the entire crane is shown in Fig. 66. The external forces are the downward pressure of the pulley, equal to 8 tons; the weights of the parts, acting through their mid-points; the horizontal reaction of the upper floor, represented by A; the horizontal reaction of the lower floor represented by B_x; and the vertical reaction of the lower floor represented by B_y. This external system is coplanar, nonconcurrent and nonparallel; appropriate conditions of equilibrium are $\Sigma M = \Sigma F_x = \Sigma F_y = 0$.

Application of these conditions, with origin of moment taken at B (in order to eliminate B_x and B_y), gives

$\Sigma M_B = (A \times 18) - (0.9 \times 11) - (1.1 \times 7) - (8 \times 20) = 0$, whence $A = 9.86$ tons;
$\Sigma F_x = -9.86 + B_x = 0$, whence $B_x = 9.86$ tons;
$\Sigma F_y = B_y - 8 - 0.8 - 0.9 - 1.1 = 0$, whence $B_y = 10.8$ tons.

Graphical solution: The resultant R (Fig. 67) of the wholly known forces is determined by the method described in Art. 29. R and the remaining forces A, B_x and B_y are in equilibrium. The special condition of equilibrium for four such forces (Art. 51)

[1] If the three unknown forces are concurrent or parallel, the problem is indeterminate.

is that the resultant of any pair balances the other pair; therefore the resultant of any two of these forces, say B_x and B_y, must balance the resultant of the other two, R and A. Since these balanced resultants must be colinear, both must act along the line through 1 (point of concurrence of B_x and B_y) and 2 (point of concurrence of A and R). The force triangle $AEFA$ is now drawn, thus determining EF $(=A)$ and FA (= resultant of B_x and B_y). Force triangle FGA then gives FG $(=B_y)$ and GA $(=B_x)$. (If any two of the unknown forces are imagined replaced by their resultant, acting through their intersection but with direction unknown, this problem becomes like (2), next discussed. The resultant can be solved for by the method illustrated in the next example, and can then be resolved into components parallel to the forces it replaced; these components are equal to the replaced forces.)

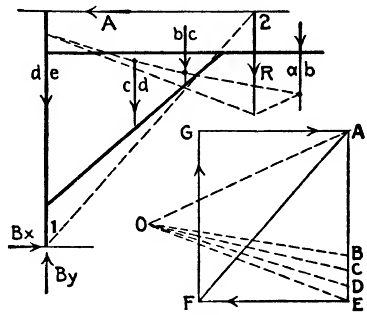

FIG. 67

(2) All the forces except two are wholly known; only the line of action of one of these two and a point in that of the other are known, and these two are to be completely determined.

EXAMPLE 2. The roof truss represented in Fig. 68 sustains two loads, one of 35,000 lbs. (weight of roof and truss) and one of 50,000 lbs. (wind pressure). The left end A of the truss is supported on rollers; the right end B is pinned. It is required to determine the reactions at A and at B.

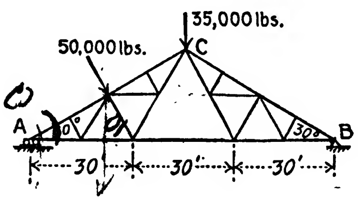

FIG. 68

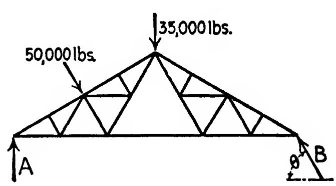

FIG. 69

Algebraic solution: The free body diagram of the truss is shown in Fig. 69. The external forces are the two loads, the reaction at A (known to be vertical because exerted by the rollers) represented by A, and the reaction at B (direction unknown) represented by B, making the angle θ to the horizontal. The external system is coplanar, nonconcurrent and nonparallel; appropriate conditions of equilibrium are $\Sigma M = \Sigma F_x = \Sigma F_y = 0$. Application of these conditions, with moment origin at B, gives

$$\Sigma M_B = (35{,}000 \times 45) + (50{,}000 \times 60 \cos 30°) - (A \times 90) = 0, \text{ whence } A = 46{,}400 \text{ lbs.};$$
$$\Sigma F_x = -B \cos\theta + 50{,}000 \sin 30° = 0; \text{ and}$$
$$\Sigma F_y = +B \sin\theta - 35{,}000 - 50{,}000 \cos 30° + 46{,}400 = 0.$$

The last two equations, solved simultaneously, give

$$B = 40{,}500 \text{ lbs. and } \theta = 51° \ 50'.$$

Graphical solution: The graphical solution is effected by drawing the force and the string polygons, making both close since the force system is in equilibrium.

The polygon ABC (Fig. 70) for the known forces is drawn and continued with a line

through C parallel to the (vertical) left-hand reaction. The end of that line, as yet unknown, is to be marked D; that point once determined then DA will represent the right-hand reaction. To find D a string polygon is constructed. The lines of action of the several forces are lettered to agree with the notation in the force polygon, a pole O is chosen, and the rays OA, OB and OC are drawn. To make use of the known point 1 of the fourth force (right-hand reaction) the string polygon must be begun at that point. The string oa is the one to draw through that point (to ab), then ob and oc are drawn as shown. The string od must pass through points 1 and 4, and so is determined. Next the ray OD is drawn parallel to the corresponding string od, and D (intersection of OD and CD) thus determined. CD and DA represent the reactions at the left and right ends of the truss respectively.

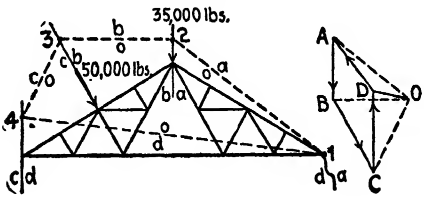

FIG. 70

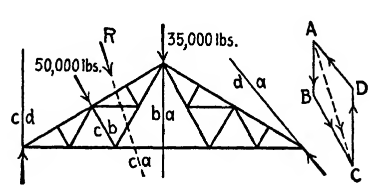

FIG. 71

The following special graphical method is simpler than the one given above. The resultant $R(= AC$, Fig. 71) of the 35,000 and 50,000 lb. forces is determined by the methods of Art. 23. R and the two unknown reactions are in equilibrium and so must be concurrent. The right reaction therefore passes through the intersection (not shown) of R and the left reaction. This establishes the direction of the right reaction and the force triangle $ACDA$ gives CD (left reaction) and DA (right reaction).

59. Noncoplanar Forces in Equilibrium.

— The conditions of equilibrium for noncoplanar forces are stated in Art. 48, 49 and 50. To show how these conditions are applied we now illustrate three typical cases.

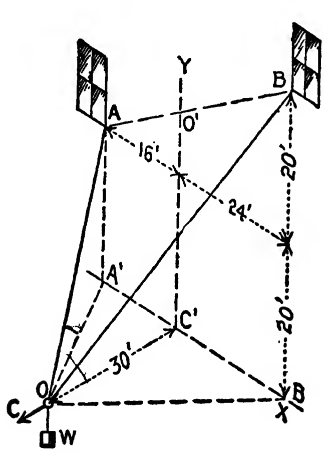

FIG. 72

(1) A system of noncoplanar concurrent forces is in equilibrium; all the forces except three are wholly known. The lines of action of these three are known and their magnitudes and senses are to be determined.

EXAMPLE 1. A heavy body W (Fig. 72) weighing 1000 lbs. is suspended from a ring over the center of a street 60 ft. wide; the ring is supported by three ropes OA, OB and OC; A and B are points on the face of a building as shown, and C is a point on the face of a building (not shown) on the opposite side of the street, OC being perpendicular to the face of the buildings. It is required to determine the tension in each rope.

Solution: The free body diagram for the ring is shown in Fig. 73. The external forces are the downward pull of 1000 lbs. and the rope pulls represented by L, M and N. The system is noncoplanar and concurrent; suitable conditions of equilibrium are $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$. Rectangular axes are chosen, the Y-axis being vertical, the

x-axis parallel to the street, and the z-axis perpendicular to the street. To get the rectangular components of L, M and N the following angles are required:

$A'OC' = \tan^{-1} A'C'/OC' = 28°$, $B'OC' = \tan^{-1} B'C'/OC' = 38° 40'$,
$AOA' = \tan^{-1} AA'/OA' = 30° 30'$, $BOB' = \tan^{-1} BB'/OB' = 46° 10'$.

The rectangular components of the unknown forces are therefore as follows:

$L_x = L \cos 30° 30' \times \sin 28° = 0.405\,L$, $N_x = 0$,
$L_y = L \sin 30° 30' = 0.507\,L$, $N_y = 0$,
$L_z = L \cos 30° 30' \times \cos 28° = 0.760\,L$, $N_z = N$.
$M_x = M \cos 46° 10' \times \sin 38° 40' = .04325\,M$,
$M_y = M \sin 46° 10' = 0.721\,M$,
$M_z = M \cos 46° 10' \times \cos 38° 40' = 0.5405\,M$.

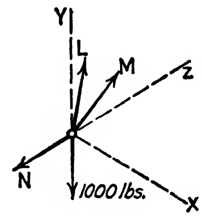

FIG. 73

The application of the conditions of equilibrium gives the equations

$$\Sigma F_x = -0.405\,L + 0.4325\,M + 0 + 0 = 0,$$
$$\Sigma F_y = +0.507\,L + 0.721\,M + 0 - 1000 = 0, \text{ and}$$
$$\Sigma F_z = -0.760\,L - 0.5405\,M + N + 0 = 0.$$

These equations, solved simultaneously, give

$$L = 846 \text{ lbs.}, \quad M = 792 \text{ lbs.}, \quad N = 1072 \text{ lbs.}$$

(2) A system of noncoplanar parallel forces is in equilibrium; all the forces except three are wholly known. The lines of action of these three are known and their magnitudes and senses are to be determined.

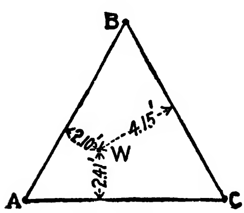

FIG. 74

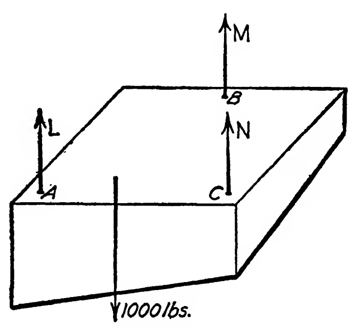

FIG. 75

EXAMPLE 2. An irregular block weighing 1000 lbs. is suspended from the ceiling of a room by means of three vertical ropes; the points of attachment at the ceiling lie at the vertices of an equilateral triangle ABC (Fig. 74) whose sides are 10 ft. long; W is the projection of the center of gravity of the block upon the ceiling. It is required to determine the tension in each rope.

Solution: The free body diagram for the block is shown in Fig. 75. The external forces are the weight of the block and the three vertical rope pulls, represented by L, M and N. The system is noncoplanar, nonconcurrent, and parallel; appropriate conditions of equilibrium are that the moment of the system about each of any three coplanar nonparallel axes perpendicular to the forces equal zero, that is, $\Sigma M_1 = \Sigma M_2 = \Sigma M_3 = 0$. Application of these conditions, with lines AB, BC and CA chosen as axes of moments (in order that two of the unknown forces may be eliminated from each moment equation), yields the following equations

$$\Sigma M_{AB} = (N \times 8.66) - (1000 \times 2.10) = 0,$$
$$\Sigma M_{BC} = (L \times 8.66) - (1000 \times 4.15) = 0,$$
and
$$\Sigma M_{CA} = (M \times 8.66) - (1000 \times 2.41) = 0.$$

Solution of these equations gives

$$L = 479 \text{ lbs.}, \; M = 278 \text{ lbs.}, \; N = 243 \text{ lbs.}$$

As a check on the results, the condition that the algebraic sum of the forces must equal zero is applied, thus:

$$\Sigma F = -1000 + 479 + 278 + 243 = 0.$$

(3) A system of noncoplanar nonconcurrent nonparallel forces is in equilibrium; all the forces except six are wholly known. The lines of action of these six are known, and their magnitudes and senses are to be determined.

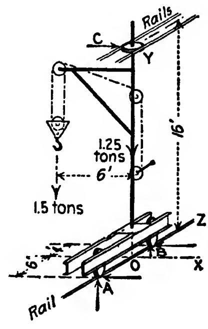

FIG. 77

EXAMPLE 3. Figure 76 shows a velocipede crane. The crane can be run along on a single rail below, tipping being prevented by two overhead rails which guide a horizontal wheel mounted on the top of the crane post. The crane weighs 1.25 tons, and it is balanced so that its center of gravity is in the axis of the post. It is required to determine the supporting forces (exerted by the rails) when the crane supports a load of 1.5 tons and the jib is swung out at right angles toward the left.

Solution: The free body diagram for the crane is shown in Fig. 77. The positions of the rails are indicated by lines to make the drawing clear, and rectangular axes are assumed, the y-axis being vertical, the x-axis perpendicular to the rails, the z-axis parallel to the rails. The external forces acting on the crane are the load and the three rail reactions, one on each wheel. Each reaction of the lower rail has an x component and a y component; the z components are zero because the rail is level

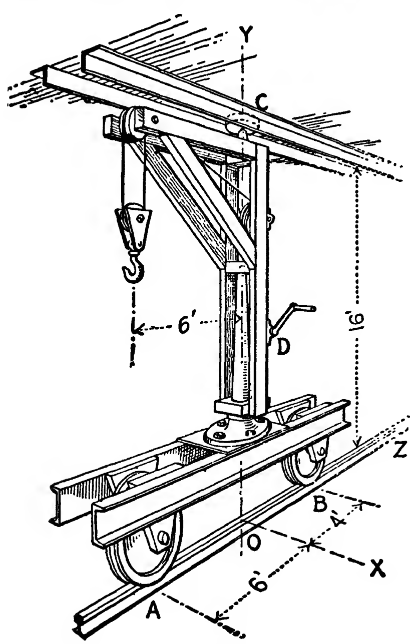

FIG. 76

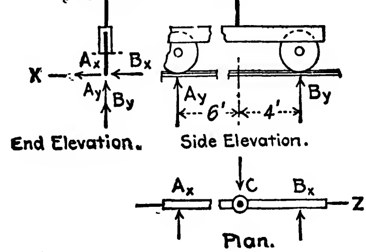

FIG. 78

and there is therefore no tendency for the crane to roll. These reactions of the lower rail are represented by their components A_x and A_y, B_x and B_y. The reaction of the upper rail is evidently horizontal and perpendicular to the rail; it is represented by C

and its sense is evidently as shown. The external system (weight of crane, load, C, A_x, A_y, B_x, B_y) is noncoplanar, nonconcurrent, nonparallel. For such a system there are, in general, six conditions of equilibrium, but this system has but five because there are no "z forces" (see free body diagram). The five conditions of equilibrium yield the following equations:

$$\Sigma F_x = A_x + B_x - C = 0, \quad \ldots (1)$$
$$\Sigma F_y = A_y + B_y - 1.25 - 1.5 = 0, \quad \ldots (2)$$
$$\Sigma M_x = (B_y \times 4) - (A_y \times 6) = 0, \quad \ldots (3)$$
$$\Sigma M_y = (B_x \times 4) - (A_x \times 6) = 0, \quad \ldots (4)$$

and
$$\Sigma M_z = (C \times 16) - (1.5 \times 6) = 0. \quad \ldots (5)$$

Solution of these equations gives

$C = .5625$ tons, $A_x = .225$ tons, $A_y = 1.10$ tons, $B_x = .3375$ tons, $B_y = 1.65$ tons.

Solution of this problem can also be effected by making use of the principle that if the forces of a system in equilibrium be represented by vectors, then the projection of the vectors on any plane represents a force system also in equilibrium (see Art. 50). Figure 78 shows such projections on the x-y, y-z, and z-x planes of Fig. 77. From the y-z projection (side elevation),

$$\Sigma M_A = B_y \times 10 - 2.75 \times 6 = 0, \quad \text{or} \quad B_y = 1.65 \text{ tons};$$

and
$$\Sigma M_B = -A_y \times 10 + 2.75 \times 4 = 0, \quad \text{or} \quad A_y = 1.10 \text{ tons}.$$

From the x-y projection (end elevation),

$$\Sigma M_A = C \times 16 - 1.5 \times 6 = 0, \quad \text{or} \quad C = .5625 \text{ tons}.$$

From the z-x projection (plan),

$$\Sigma M_A = -B_x \times 10 + .5625 \times 6 = 0, \quad \text{or} \quad B_x = .3375 \text{ tons};$$

and
$$\Sigma M_B = -A_x \times 10 + .5625 \times 4 = 0, \quad \text{or} \quad A_x = .225 \text{ tons}.$$

CHAPTER IV

SIMPLE STRUCTURES

§ 1. Simple Frameworks; Truss Type

60. Definition and Description. — A framework consisting of straight members, with axes all lying in the same plane, so connected as to form a triangle or series of triangles, is called a *truss*. Such a framework is rigid under loads applied at the joints and in the plane of the members, and it will be seen presently that in the usual type of construction the members are subject only to longitudinal tension or compression as a result of such loading. Figure 79 represents a simple truss of a common type.

In order to make the axes of all members lie in one plane, and the truss symmetrical with respect to that plane, some of the members must be made in parts or with forked ends. For example, see Fig. 80, which shows plan and elevation of joint C of the truss outlined. Here four members are pinned together, one vertical V (double), one diagonal D (single) and two horizontals H_1 and H_2 (each double).

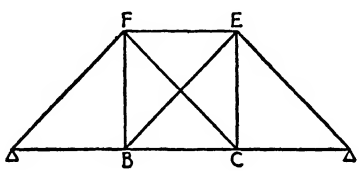

Fig. 79

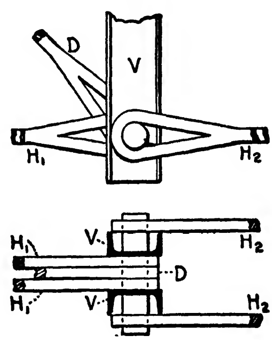

Fig. 80

Trusses are used in roof construction, for bridges, and in general wherever loads must be carried over a wide span or at points remote from primary support. They are commonly constructed of wood, of steel, or of a combination of wood and steel. Wooden members are usually rectangular and are nailed or bolted together with more or less mortising. Steel members are generally made up of " structural shapes " (angles or channels) and plates, and are connected by pins as shown in Fig. 80, or riveted together.

In this (first) section of the chapter, it is assumed that: (i) the members of each truss are connected to one another at their ends only; (ii) they are connected by smooth pins perpendicular to the plane of the truss; (iii) the loads and reactions are applied on the pins only, and in such manner that the line of action of each cuts the axis of the pin on which it is

applied; and (iv) the weight of each member, unless neglected, is replaced by two components (each one-half the weight) at the ends of the member. It follows from these assumptions that each member is subjected to only two forces (the pin pressures at its ends), and that these two forces, being in equilibrium, are equal, opposite and act along the axis of the member.

61. Stress in a Member. — As just explained, in the preceding article, the forces exerted on any member are equal pulls (Fig. 81) or equal pushes (Fig. 82) directed along the axis of the member.

Any two parts of a member as m and n (Fig. 81 or 82) exert forces upon each other which are equal, opposite, and colinear (Art. 7). Let A denote

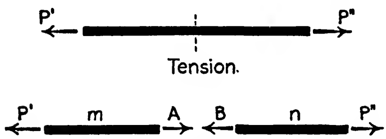

FIG. 81

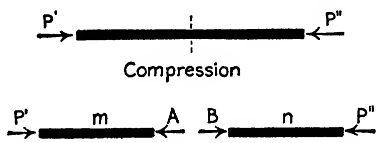

FIG. 82

the force exerted on m by n, and B the force exerted on n by m. Since m and n are at rest under the action of two forces only, it follows that A balances P' and B balances P''. Therefore, (i) A and B each equals the end pull or push as the case may be; (ii) A and B act along the axis of the member; and (iii) A and B are pulls if the end forces are pulls (Fig. 81) and they are pushes if the end forces are pushes (Fig. 82). It should be noted that these results (i, ii and iii) are independent of the relative lengths of m and n; that is they hold for any imagined plane of separation.

By "stress in a member" of a truss we mean either of the forces (A or B) which two parts of the member on opposite sides of any (imagined) cross-section exert upon each other. "Amount or magnitude of the stress" means the magnitude of either of the forces mentioned. Thus to determine the amount of the stress in any particular instance, one has only to determine in some way, the magnitude of A or B. "Kind of stress" refers to the pull or push aspect of the stress. If A and B are pulls (Fig. 81), then the stress is said to be tensile; if they are pushes (Fig. 82), the stress is said to be compressive. Thus to determine the kind of stress in a member, one has only to determine in some way whether A or B is a pull or push.

62. Determination of Stress in a Member of a Loaded Truss. — An explanation of the general method by which the stress in a truss member is determined will be more readily understood if preceded by a simple illustration.

Suppose it is desired to find the stress in member CD of the truss shown in Fig. 83. Imagine the members CD, BD and BE cut so as to separate that portion of the truss to the right of the line 1-1 from the remainder of the structure. (This cutting off of a part of a truss in order to isolate

it is sometimes called " passing " a section; here we pass the section 1-1.) The portion of the truss thus isolated, shown in Fig. 84, is in equilibrium under the external forces that act on it. Now these external forces are

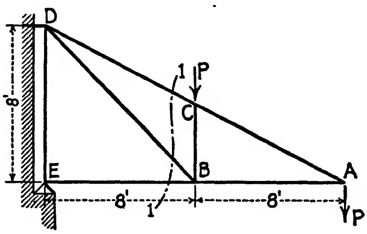

FIG. 83

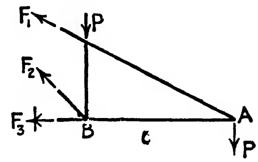

FIG. 84

the loads (P = 1000 lbs.) and the forces exerted on the cut ends of the portions of CD, BD and BE; these latter forces are the stresses in the respective members and so are axial. Let it be assumed[1] that these unknown stresses are all tensions, and let them be accordingly represented on Fig. 84 by the pulls F_1, F_2 and F_3; Fig. 84 then becomes a free body diagram for the isolated portion of the truss. The system of external forces acting is readily solved for F_1, thus:

$$\Sigma M_B = -(1000)(8) + (F_1)(3.58) = 0, \text{ whence } F_1 = +2230 \text{ lbs.}$$

The plus sign indicates that the sense of F_1 is as assumed; therefore F_1 is a pull and the stress in CD is 2230 lbs. tension. If the stresses in BD and BE are desired they may readily be found by solving for F_2 and F_3, using the conditions

$$\Sigma F_x = 0, \text{ and } \Sigma F_y = 0.$$

These equations solved simultaneously give $F_2 = +1410$ lbs. and $F_3 = -3000$ lbs.; therefore F_2 is a pull as assumed and F_3 is a push (as indicated by the crossed arrow-head), and the stresses in BD and BE are respectively 1410 lbs. tension and 3000 lbs. compression.

From the above example it is seen that the problem of determining the stress in a truss member is solved essentially like any other problem in equilibrium, namely, by choosing a body for consideration, drawing the free body diagram, and applying appropriate conditions of equilibrium to

[1] It is not difficult to see, from the arrangement of the members and loads shown here that the stress in DC is tension and in BE compression, and so it might be said that the kind of stress in these members is *judged* rather than assumed. The kind of stress in BD is not so easily perceived, and so is simply guessed at or assumed. It will be recalled that when, in the algebraic solution of problems, a force is encountered whose sense is unknown, a sense is assumed and the equilibrium equations written accordingly. If, upon solving for the force, a positive result is obtained, the sense of the force is as assumed; if a negative result is obtained, the sense of the force is opposite to that assumed. An error in assuming the kind of stress in a truss member is similarly discovered, and so such errors are really of no consequence so far as the solution of the problem is concerned.

the external forces. The body chosen for consideration is a portion of the truss cut off or isolated from the remainder by passing a section as in the example given, and must be such a portion that (i) the desired stress is one of the external forces acting, and (ii) the system of external forces acting is solvable for the desired stress.

The system of external forces acting on such an isolated portion of a truss consists of the reaction and loads (if any) on that part and the stresses exerted on that part by the other (adjoining) part. The loads will be regarded as completely given or known. The reactions can be computed by methods already explained. The lines of action of the stresses coincide with the axes of the respective members as proved in Art. 60; their magnitudes and "kinds" are generally unknown at the outset. In an algebraic solution of such an external system, it is necessary to assume senses (arrow-heads) for the stresses. As pointed out in the footnote (p. 60) erroneous assumptions are of no consequence, but for the sake of practice the student is advised to *judge* the kind of stress by inspection of the entire truss diagram or the part under consideration and to place arrow-heads in the free body diagram accordingly (see Art. 54). For tension, the stress is a *pull* and the arrow points *away from* the part; for compression the stress is a *push* and the arrow points *toward* the part.

63. Analysis of a Truss. — By analysis of a truss is meant the determination of the amount and kind of stress in each member caused by stated loads on the truss. One might determine the stresses in any random order, and thus accomplish the analysis. But it is advantageous to proceed after some definite plan. One such is explained in the following article. In many cases short cuts or special methods would be employed by one experienced in truss analysis. For these the student is referred to special works on that subject.

64. Joint to Joint Plan. — By "joint of a truss" we mean a pin and the adjoining short parts of the members it connects. A joint is isolated by passing a section around the pin and cutting all members meeting there. The external forces on such a joint consist of the load or reaction (if any) at the joint and the stresses (pushes or pulls) on the ends of the cut members. We call these the "forces at the joint."

The set of forces at a joint are concurrent and coplanar. Such a set is solvable only if it has not more than two unknowns. Hence to analyze a truss on this plan, proceed as follows: (i) Look for a joint at which there are not more than two unknown forces. At the outset, there may be no such joint. If this is the case, determine one or all of the reactions of the supports of the truss; that done, then a joint meeting this requirement can be found. (ii) Solve the system of forces at that joint for the unknowns. (iii) Repeat item (ii) again and again until all the stresses have been determined. (iv) It may not be possible to repeat at *all* the joints. In such case, after having considered as many joints as possible in the manner

just explained, determine by the general method (Art. 62) any one of the remaining unknown stresses. Then it will be possible generally to proceed "by joints" to the complete analysis.

EXAMPLE 1. Figure 85 represents a truss supported at each end; the angles equal 60°; it sustains two loads of 2000 lbs. each and one of 1000 lbs. It is required to determine the stress in each member of the truss.

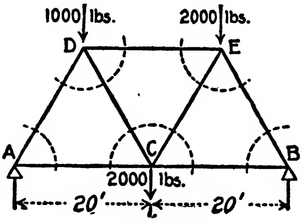

FIG. 85

Solution. The reactions A and B are determined by the methods of Art. 57. Equating the moment of the external system about B to zero gives $A = 2250$ lbs.; equating the moment of the system about A to zero gives $B = 2750$ lbs.

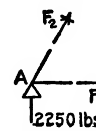

FIG. 86

At joint A there are but two unknown forces, the stresses in AD and AC. The joint is isolated (Fig. 86) and the unknown stresses represented by F_1 and F_2, both being assumed to be pulls. Then

$$\Sigma F_y = 2250 + F_2 \sin 60° = 0, \text{ whence } F_2 = -2600 \text{ lbs.}$$

The negative sign shows that F_2 is a push instead of a pull; therefore the stress in AD is compression.

$$\Sigma F_x = -2600 \cos 60° + F_1 = 0, \text{ whence } F_1 = +1300 \text{ lbs.}$$

The positive sign shows that F_1 is a pull, as assumed; therefore the stress in AC is tension.

At joint B there are but two unknown forces. The joint is shown in Fig. 87, the unknown stresses being represented by F_3 and F_4. Solution, effected as before, gives

$$F_3 = +1588 \text{ lbs. (tension)}, \quad \text{and} \quad F_4 = -3177 \text{ lbs. (compression)}.$$

Any one of the remaining joints might next be considered, since there are but two unknown forces at each. Joint C is shown in Fig. 88. The stress in AC having been found to be 1300 lbs. tension, the force exerted by the part of that member not shown

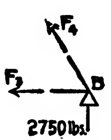

FIG. 87

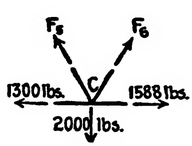

FIG. 88

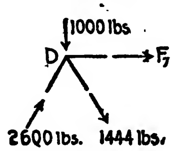

FIG. 89

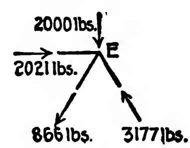

FIG. 90

upon the part shown is a *pull* of 1300 lbs. Similarly, the force exerted by the part of CB not shown upon the part shown is a pull of 1588 lbs. The unknown stresses in CD and CE are represented by F_5 and F_6. Solution gives

$$F_5 = +1444 \text{ lbs. (tension)}, \quad \text{and} \quad F_6 = +866 \text{ lbs. (tension)}.$$

Joint D is shown in Fig. 89. The stress in AD having been found to be 2600 lbs. compression, the force exerted by the part of that member not shown upon the part shown in a *push* of 2600 lbs. The stress in DC having been found to be 1444 lbs. tension, the force exerted by the part of that member not shown upon the part shown is a *pull* of 1444 lbs. The unknown stress in DE is represented by F_7. Solution gives

$$F_7 = -2021 \text{ lbs. (compression)}.$$

All the stresses are now known, but as a check joint E (Fig. 90) is considered. The forces acting, are the load and the stresses in the several members, all of which have

been determined and which act as shown. It is found that $\Sigma F_x = \Sigma F_y = 0$, and this affords a check upon the preceding computations.

The student is urged to always record the values obtained for the stresses on the corresponding members in the sketch of the truss, indicating tension by T and compression by C. When this is done the sketch shows at a glance the results of the solution.

EXAMPLE 2. The truss shown in Fig. 91 is supported at either end and sustains five vertical loads of 800 lbs. and one of 1200 lbs. It is required to determine the stress in each member.

Solution: The reactions R_1 and R_2 are solved for and found to be 2800 lbs. and 2400 lbs. respectively.

At joint A there are but two unknown stresses; solution by the methods illustrated in Ex. 1 gives for the stress in AB, 3960 lbs. compression, and for the stress in AH, 2800 lbs. tension.

At joint G there are but two unknown stresses; solution gives for the stress in GF, 3400 lbs. compression, and for the stress in GI, 2400 lbs. tension.

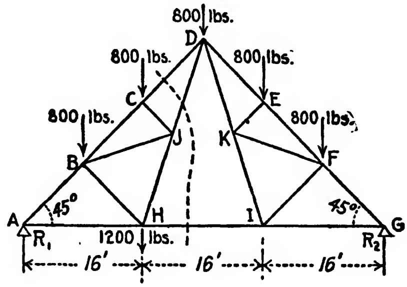

FIG. 91

No joint remains at which there are but two unknown stresses, therefore it is impossible to proceed further by the method used heretofore. Obviously, if the stress in almost any one of the remaining members could be ascertained, the solution by the joint-to-joint method could be resumed; thus if the stress in HI were known it would be possible to solve for the other stresses at joint H. To determine the stress in HI the general method of Art. 62 is employed. A section is passed cutting CD, JD and HI as shown. The free body diagram for that part of the truss to the left of this section is shown in Fig. 92; the external forces are the reaction at A, the three loads, and the stresses in the cut members, represented by S_1, S_2 and S_3. By taking moments about D (the intersection of S_2 and S_3), S_1 is found to be 1600 lbs. tension.

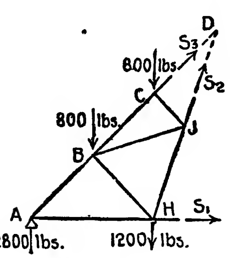

FIG. 92

It is now possible to solve at joint H, then at joint B, then at C (or J), then at D, then at E, then at K, and then at F. All the stresses will then have been determined, but as a check it can be ascertained if the forces at I satisfy the conditions

$$\Sigma F_x = \Sigma F_y = 0.$$

65. Solution by Inspection. — After a certain amount of practice in the analysis of trusses the student will find that sometimes the stress in members can be determined simply by inspection, and that often the *kind* of stress and the relative magnitude of different stresses can be so determined. As examples, let the three trusses shown in Figs. 93, 94 and 95 be considered. For the truss of Fig. 93 the reactions are each

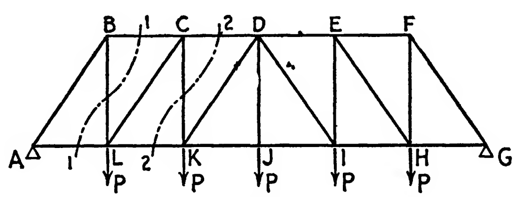

FIG. 93

2.5 P. If section 1–1 is passed and the left portion of the truss considered, it is seen that there are only two vertical forces (or forces with vertical components) acting, namely, the reaction 2.5 P up and the stress in BL, which must then be equal and opposite to the reaction. Therefore the stress in BL is 2.5 P tension. In the same way on passing section 2–2 the stress in CK is seen to be 1.5 P tension, while the stresses in CD and KL are seen to be equal and of opposite kinds, since they are the only horizontal forces acting on that part of the truss to the left of 2–2. If the joint at A is isolated it is seen that the vertical component of the stress in AB balances the upward reaction and that therefore the stress in AB is compression, and is greater than 2.5 P.

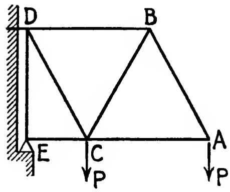

FIG. 94

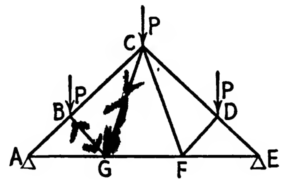

FIG. 95

For the truss of Fig. 94 it is evident that at joint A the vertical component of the stress in AB balances P, and that therefore this stress is tension and is greater than P. At joint B it is evident that the stresses in AB and BC are the only forces with vertical components, that these vertical components must therefore be equal and opposite, and that as the members have the same inclination to the horizontal their stresses must be equal in magnitude and opposite in kind.

For the truss of Fig. 95 one may imagine the joint B isolated and forces summed up along BG (perpendicular to AB–BC). Evidently the stress in BG is compression, and being equal to the component of P perpendicular to AB–BC, is less than P. Again, if the joint at G is isolated and forces summed up along the vertical, it is seen that the stress in GC is tension, an upward pull balancing the downward push of BG. And since the vertical components of the stresses in these two members are equal, the more nearly horizontal member will have the greater stress.

Such analysis by inspection develops an understanding of truss action, and is of great value as a means of checking the results of a formal solution. The student is advised to practice it, comparing the results so obtained with those found by a complete algebraic or graphical analysis.

66. Graphical Analysis of Trusses. — Joint to Joint Plan. — As in the corresponding algebraic solution, we isolate a portion of the truss by passing a section around a joint, cutting the members that are there connected. The external forces that act on the body thus isolated are then solved for by graphical instead of algebraic methods.

The notation for graphical work described in Art. 12 can be advantageously systemized as follows: Each triangular space in the truss diagram is marked by a lower case letter, also the space between consecutive lines

of action of the loads and reactions; then the two letters on opposite sides of any line serve to designate that line, and the same capital letters are used to designate the magnitude of the corresponding force and its vector. This scheme of notation is a great help in graphical analyses of trusses. It is used in the following example.

EXAMPLE. The truss shown in Fig. 96 is supported at either end and sustains three loads of 1000 lbs. and two loads of 500 lbs. It is required to determine the stress in each member.

Solution: The reactions are determined by inspection; it is evident that each is equal to one-half the total load, or 2000 lbs. The truss is then lettered in accordance with the directions of Art. 66.

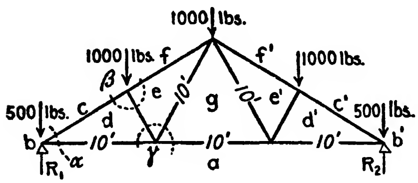

FIG. 96

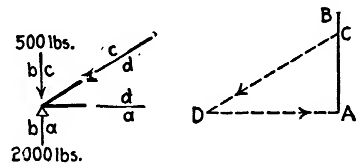

FIG. 97

Section α is passed; the joint thus isolated is shown in Fig. 97. The external forces are the reaction, the two loads, and the unknown stresses in *cd* and *ad*. The force polygon for the joint is constructed, the known vectors *AB* and *BC* being drawn first; from *C* a line is drawn parallel to *cd*; from *A* a line is drawn parallel to *ad*; these lines determine *D* and complete the closed polygon *ABCDA*. *CD* (3000 lbs.) represents the stress in *cd*; from the force polygon it is apparent that *CD* is a push, therefore the stress in member *cd* is 3000 lbs. compression. *DA* (2600 lbs.) represents the stress in *da*; from the force polygon it is apparent that *DA* is a pull, therefore the stress in *da* is 2600 lbs. tension.

Next, section β is passed; the joint thus isolated is shown in Fig. 98. The external forces are the stress in *cd* (3000 lbs. compression), the load, and the unknown stresses in *de* and *ef*. The force polygon is drawn in the following order: *DC*, *CF*. a line from *F*

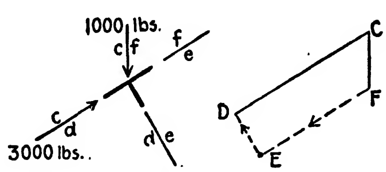

FIG. 98

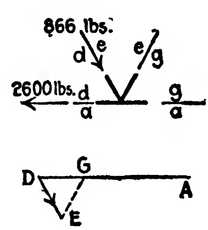

FIG. 99

parallel to *fe*, a line from *D* parallel to *de*. These last two lines determine *E* and complete the closed polygon *DCFED*. *FE* (2500 lbs.) represents the stress in *fe*; from the force polygon it is seen to be compression. *ED* (866 lbs.) represents the stress in *ed*; from the force polygon it also is seen to be compression.

Next, section γ is passed; the joint thus isolated is shown in Fig. 99. The external forces are the stress in *da* (2600 lbs. tension), the stress in *ed* (866 lbs. compression) and the unknown stresses in *eg* and *ga*. The force polygon is drawn in the following order:

AD, *DE*, a line from *E* parallel to *eg*, a line (already drawn) from *A* parallel to *ag*. These last two lines determine *G* and complete the closed polygon *ADEGA*. *EG* (866 lbs.) represents the stress in *eg*, and *GA* (1732 lbs.) the stress in *ga*. From the force polygon both are seen to be tensions.

On account of the symmetry of the truss and loading, the forces in the remaining members are now known.

In drawing the force polygon for all the external forces on the part of a truss included within a section about a joint, it will be advantageous to represent the forces in the order in which they occur about the joint. A force polygon so drawn will be called a polygon for the joint; and for brevity, if the order taken is clockwise the polygon will be called a clockwise polygon, and if counter-clockwise it will be called a counter-clockwise polygon. *ABCDA* (Fig. 97) is a clockwise polygon for joint *b* of Fig. 96. The student should draw the counter-clockwise polygon for the joint, and compare with *ABCDA*.

67. Stress Diagrams. — If the polygons for all the joints of a truss are drawn separately as in the preceding illustration, then the stress in each member will have been represented twice. It is possible to combine the polygons so that it will not be necessary to represent the stress in any member more than once, thus reducing the number of lines to be drawn. Such a combination of force polygons is called a *stress diagram*. Figure 100 is a stress diagram for the truss of Fig. 96 loaded as there shown. Comparing the part of the stress diagram consisting of solid lines with Figs. 97, 98 and 99, it is seen to be a combination of the latter three figures. It will also be observed that the polygons are all clockwise polygons; counter-clockwise polygons also could be combined into a stress diagram.

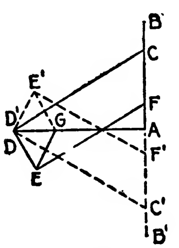

FIG. 100

Directions for constructing a stress diagram for a truss under given loads:

(1) Letter the truss diagram as already explained.

(2) Determine the reactions. (In some exceptional cases this stage may or must be omitted; also stage (3). See Art. 68 for two illustrations.)

(3) Construct a force polygon for all the external forces applied to the truss (loads and reactions), representing them in the order in which their points of application occur about the truss, clockwise or counter-clockwise. (The part of that polygon representing the loads is called a load line.)

(4) On the sides of that polygon construct the polygons for all the joints. They must be clockwise or counter-clockwise ones, according as the polygon for the loads and reactions was drawn clockwise or counter-clockwise. The first polygon drawn must be for a joint at which but two members are fastened; the joints at the supports are usually such. Next

the polygon is drawn for a joint at which not more than two stresses are unknown; that is, of all the members fastened at that joint the forces in not more than two are unknown. Then the next joint at which not more than two stresses are unknown is considered.[1]

The above directions do not provide for cases in which more than two of the stresses at a joint are unknown. We meet this difficulty as explained in Art. 64, solving for one or more stresses, as may be necessary, by the general method there mentioned and then using the values thus determined to complete the stress diagram.

EXAMPLE. The truss shown in Fig. 101 below is supported at either end and sustains seven loads of 1000 lbs. and two of 500 lbs. It is required to determine the stress in each member.

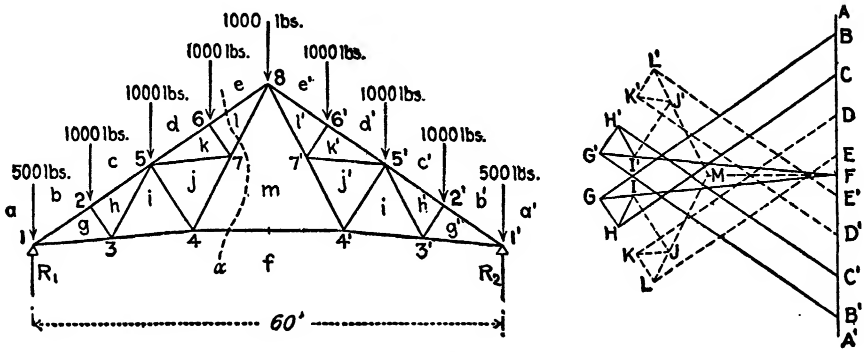

FIG. 101

Solution: The reactions are determined by inspection; it is evident that each is equal to one-half the total load, or 4000 lbs.

The truss is then lettered and the polygon (clockwise in this case) $ABCDEE'D'C'B'A'FA$ is constructed for the loads and reactions.

Joint 1 is next considered; there are but two unknown stresses and the (clockwise) force polygon $FABGF$ is readily drawn. Similarly, for joint 2 the polygon $GBCHG$ and for joint 3 the polygon $FGHIF$ are drawn. The polygons for joint 1′, 2′ and 3′ are respectively $B'A'FG'B'$, $C'B'G'H'C'$, and $H'G'FI'H'$.

No joint remains at which there are but two unknown stresses, therefore in order to proceed with the solution it is necessary (as in Ex. 2, Art. 64) to determine in some way the stress in some one of the remaining members. Thus if the stress in mf could be found the force polygon for joint 4 could be drawn. To determine the stress in mf the section α is passed and the left part of the truss considered. By taking moments about point 8 (intersection of el and lm) the stress S in mf is easily found, thus

$$\Sigma M_8 = -(4000 \times 30) + (500 \times 30) + (1000 \times 22.5) + (1000 \times 15) + (1000 \times 7.5) + (S \times 17.5) = 0, \text{ whence } S = 3425 \text{ lbs. (tension).}$$

[1] The student is urged to make sketches of the bodies (parts of truss) upon which the forces, whose polygons are being drawn, act. That is, a free body diagram should be constructed for each joint, as in the example of Art. 66. A force acting upon the "cut" end of a member and toward the joint is a push, and the stress in the member is compressive; if the force acts away from the joint, it is a pull, and the stress is tensile.

The stress S is then represented in its proper place on the stress diagram as MF and the force polygon for joint 4 is drawn; it is $MFIJM$. Completion of the solution presents no difficulties.

68. Special Cases. — In some cases the reactions cannot be determined in advance. The stress diagram can still be drawn if the frame under consideration is statically determinate,[1] the polygons being drawn in the usual manner for all joints save those at which the unknown reactions are applied. This will give all the stresses; the reactions can then be found by drawing the polygons for the joints at which they act.

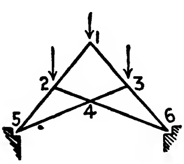

FIG. 102

Figure 102 represents such a case, the frame being pinned to its supports. The diagram can be constructed by drawing in succession the proper polygons (all clockwise or counter-clockwise) for joints 1, 2, 3 and 4. Then, if desired, the reactions can be determined by drawing the polygons for joints 5 and 6.

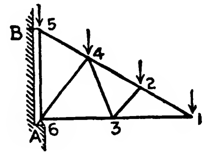

FIG. 103

Figure 103 represents a case where the reactions can be determined at stage (2) of the analysis, but determination of the reactions is not essential for the construction of the stress diagram. The truss is supported by a shelf A and a tie B. The stress diagram can be constructed by drawing in succession proper polygons for joints 1, 2, 3, 4 and 5. The reaction at B is determined by the polygon for joint 5; that at A by the polygon for joint 6.

§ 2. Simple Frameworks; Crane Type

69. Description. — The frames here considered, like the trusses which have been discussed, are plane, and symmetrical with respect to the plane of the frame. Again, like the trusses, they are assumed to be made of members connected by frictionless pins; thus each pin pressure lies in the plane of the frame, and the line of action cuts the axis of the pin. Unlike the trusses, these frames may include members which are pinned to others at more than two points, and the loads are applied anywhere,

[1] A truss the stresses in which can be found by the methods of statics is called *statically determinate.* As stated in Art. 60, a framework is rigid if the members that compose it form a triangle or series of triangles; additional members, not essential for rigidity, are said to be redundant, and introduce by their presence a greater number of unknowns into some of the force systems involved in the analysis of the frame than there are independent conditions of equilibrium. It is therefore not possible to determine all the stresses by the methods of statics, and such frames are therefore called *statically indeterminate.* Trusses (and other structures) may also be statically indeterminate because of redundant supports, either of the whole or of its parts. For a full discussion of this subject, readers are referred to standard works on structures, such as Johnson, Bryan, and Turneaure's *Modern Framed Structures.*

instead of at the joints only. The result of these conditions is that some or all of the members are subject not merely to tension or compression, but also to bending. We do not, therefore, attempt to determine the stresses in the members of these frames, but limit the solution to a determination of the forces (pin pressures, reactions of supports, etc.) that act on each member.

70. Analysis of a Crane. — The analysis of a crane consists in the determination of every force (magnitude and direction) acting on each part or member, due to the weight of the crane or to the applied loads or both. Solution is effected by regarding the whole crane or parts of it as bodies in equilibrium and applying the appropriate conditions of equilibrium to the external forces acting thereon. There is opportunity for the exercise of judgment in determining on the sequence in which the parts are to be considered. Usually it will be found advantageous to consider first the crane as a whole, proceeding then to a consideration of single members or groups of members.

It must be kept in mind that the members of a crane are not, in general, two-force members as in the case of a truss, and that therefore the pressure of a pin on a member does not, in general, act along the axis of the member. Usually, its direction is unknown, and it is convenient in an algebraic solution to represent it by rectangular components with senses assumed. It should be remembered that at a pinned joint the forces exerted on the members there connected are exerted *by the pin*, and not by the members on each other. At a joint where only two members are pinned together it is permissible to regard the forces as exerted directly by the members on each other, because the pin simply transmits the pressure from one to the other. But at a joint where more than two members are pinned together the force on each member should be regarded as exerted by the pin, and the pin itself should be considered as a body in equilibrium under the action of the forces exerted on it by the members it connects.

When a crane has been analyzed, the results of the analysis should be presented by showing, on a separate sketch (free body diagram) of each individual member, the forces that act on that member, with their values.

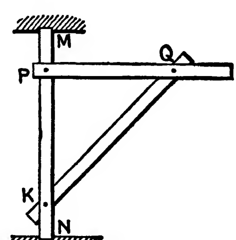

FIG. 104

EXAMPLE 1. Figure 104 represents a crane, consisting of a vertical post, horizontal boom, and inclined brace; it is supported at M and N by sockets in the ceiling and floor. The crane supports a load of 8 tons on the boom; the weights of the members may be neglected. It is required to completely analyze the crane.

Solution: The crane as a whole is first considered. The free body diagram for the whole structure is shown in Fig. 105; the external forces are the load, the horizontal reaction M of the ceiling, and the reaction of the floor (direction unknown) represented by its rectangular components N_x and N_y. (The senses of M, N_x and N_y are assumed.)

Solution of this system falls under Art. 58; it is effected as follows:

$$\Sigma M_N = -(8 \times 16) + (M \times 18) = 0, \text{ whence } M = 7.11 \text{ tons};$$
$$\Sigma F_y = -8 + N_y = 0, \text{ whence } N_y = 8 \text{ tons};$$
and
$$\Sigma F_x = -7.11 + N_x = 0, \text{ whence } N_x = 7.11 \text{ tons}.$$

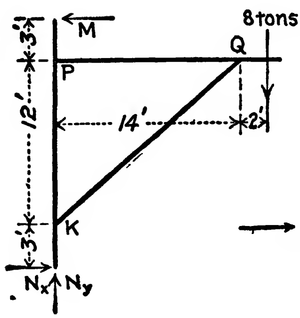

FIG. 105

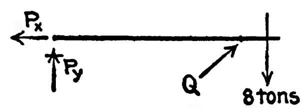

FIG. 106

FIG. 108

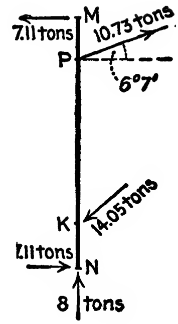

FIG. 107

Of the several parts of the crane, either the post or the boom should next be considered, because none of the forces acting on the brace is completely known. The boom is chosen and the free body diagram constructed (Fig. 106). The external forces are the load; the force Q exerted by the brace (assumed to be a push and known to act in the direction of the brace because the latter is a two-force member); and the force (direction unknown) exerted by the post at P, represented by its rectangular components P_x and P_y, with senses assumed. Solution of this system falls under Art. 58. Noting that the angle $PQK = \tan^{-1} 12/14 = 40° \, 36'$ and that the perpendicular distance from P to $KQ = 14 \sin 40° \, 36' = 9.11$ ft. the following equations may be written:

$$\Sigma M_P = -(8 \times 16) + (Q \times 9.11) = 0, \text{ whence } Q = 14.05 \text{ tons};$$
$$\Sigma F_x = 14.05 \cos 40° \, 36' - P_x = 0, \text{ whence } P_x = 10.67 \text{ tons};$$
and
$$\Sigma F_y = -8 + 14.05 \sin 40° \, 36' + P_y = 0, \text{ whence } P_y = -1.14 \text{ tons}.$$

The negative sign indicates that P_y acts down, instead of up as assumed. Compounding P_x and P_y, $P = (10.67^2 + 1.14^2)^{\frac{1}{2}} = 10.73$ tons, and the inclination of P with the horizontal is $\tan^{-1} (1.14 \div 10.67) = 6° \, 10'$. By the principle of action and reaction, all remaining forces are now readily found. Those on the post are represented on the free body diagram for that member, Fig. 107.

EXAMPLE 2. Figure 109 represents an hydraulic crane. It consists of a hollow post MN (up into which a piston can be projected) a boom PQ and a pin-connected frame KPQ. A single roller is mounted on the pin K, and two on the pin P, so that as the piston moves the frame moves with it, all rollers rolling on the post. Thus there are twelve parts: a post, a boom, two struts KP (one on each side of the post), two ties KQ (one on each side), a pin at P, one at Q, one at K, two rollers at P and one at K. The load will be taken as 10 tons and x as 15 ft.; the weights of the parts may be neglected. It is required to completely analyze the crane.

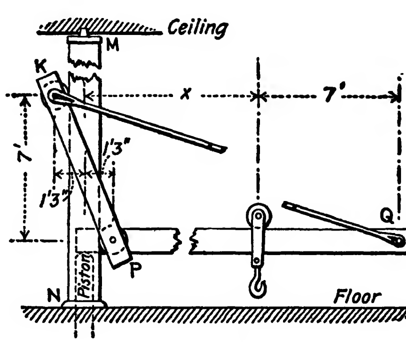

FIG. 109

Solution: The crane as a whole is first considered. The free body diagram for the entire structure is shown in Fig. 110; the external forces are the load, the pressure of the piston L, the horizontal reaction of the ceiling M, and the reaction (direction unknown) of the floor, represented by its rectangular components N_x and N_y. Partial solution of this force system is effected as follows:

$$\Sigma M_N = -(10 \times 15) + (M \times h) = 0,$$

whence $M = 150/h$ tons (h is height of post)

and $\Sigma F_x = -M + N_x = 0$, whence $N_x = M$.

L and N_y cannot be found from consideration of this force system.

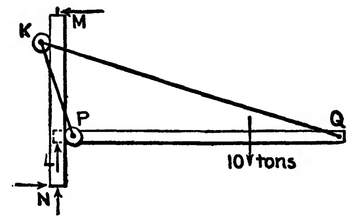

Fig. 110

Next, the frame together with the rollers is considered. The free body diagram is shown in Fig. 111; the external forces are the load, the pressure of the piston L, and the horizontal reactions R_1 and R_2 of the post against the rollers. Solution of this system falls under Art. 47; it is effected as follows:

$$\Sigma F_y = 0 \text{ gives } L = 10 \text{ tons},$$
$$\Sigma F_x = 0 \text{ shows } R_1 = R_2,$$

and $$\Sigma M = 0 \text{ gives } R_1 = 21.4 \text{ tons} = R_2.$$

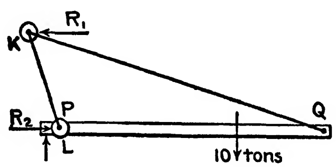

Fig. 111

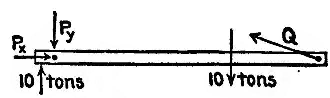

Fig. 112

Next, the boom alone is considered. The free body diagram is shown in Fig. 112; the external forces are the load, the piston pressure, the pull Q of the tie KQ (acting along the tie because the latter is a two-force member), and the pressure (direction unknown) of the pin at P, represented by its components P_x and P_y. Solution of this system falls under Art. 58; it is effected as follows:

$$\Sigma M_P = 0 \text{ gives } Q = 25.2 \text{ tons},$$
$$\Sigma F_x = 0 \text{ gives } P_x = 24 \text{ tons},$$

and $$\Sigma F_y = 0 \text{ gives } P_y = 7.2 \text{ tons}.$$

Next the roller at K is considered. It is apparent that this roller is acted on by two forces, one exerted by the post, which acts to the left and is equal to 21.4 tons, and one exerted by the pin, which must act to the right and be equal to 21.4 tons.

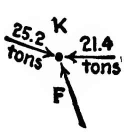

Fig. 113

Next the pin at K is considered. The free body diagram is shown in Fig. 113; the external forces are the pull of the ties (25.2 tons), the pressure of the roller (21.4 tons), and the force F exerted by the struts (known to be along the axis of the struts because each is a two-force member.) For this system,

$$\Sigma F_y = 0 \text{ gives } F = 7.8 \text{ tons}.$$

All remaining forces acting on the several parts can be found by the principle of action and reaction and by consideration of two-force members. The student should draw the free body diagrams for the ties, the struts, the post, the rollers, and the pin at P.

EXAMPLE 3. The members of the crane described in Ex. 1 have weights as follows: post, 0.8 tons; boom, 0.9 tons; brace, 1.1 tons. The load is, as in Ex. 1, 8 tons and is applied on the boom 16 ft. out from the axis of the post. The length of the boom is 22 ft.; the weights of the members may be considered as acting at their respective mid-points. It is required to completely analyze the crane, taking into account the weights of the members.

Solution: The entire structure is first considered. The free body diagram is shown in Fig. 114; solution for the unknown forces is effected as follows:

$$\Sigma M_N = 0 \text{ gives } M = 8.09 \text{ tons},$$
$$\Sigma F_x = 0 \text{ gives } N_x = 8.09 \text{ tons},$$
and $$\Sigma F_y = 0 \text{ gives } N_y = 10.8 \text{ tons}.$$

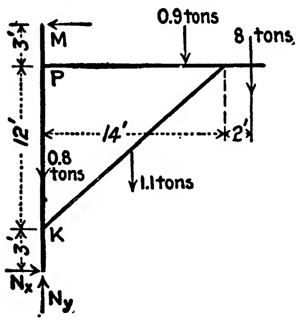

FIG. 114

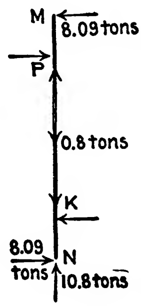

FIG. 115

Next the post alone is considered. The free body diagram is shown in Fig. 115; the external forces are the weight; the reactions M, N_x and N_y; the force (direction unknown) exerted by the boom at P, represented by its components P_x and P_y, and the force exerted by the brace at K, represented by its components K_x and K_y (the direction of the force at K is *not* along the axis of the brace, because the latter is not a two-force member when its weight is considered). Partial solution of this force system is effected as follows:

$$\Sigma M_K = 0 \text{ gives } P_x = 12.13 \text{ tons},$$
and $$\Sigma F_x = 0 \text{ gives } K_x = 12.13 \text{ tons}.$$

P_y and K_y cannot be determined from a consideration of this force system.

Next the boom alone is considered. The free body diagram is shown in Fig. 116; the external forces are the weight, the load, the pressure of the post (one component known) and the force (direction unknown) exerted by the brace at Q, represented by its components Q_x and Q_y. Solution is effected as follows:

$$\Sigma F_x = 0 \text{ gives } Q_x = 12.13 \text{ tons},$$
$$\Sigma M_Q = 0 \text{ gives } P_y = 0.95 \text{ tons},$$
and $$\Sigma F_y = 0 \text{ gives } Q_y = 9.85 \text{ tons}.$$

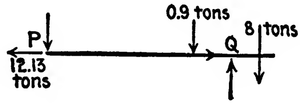

FIG. 116

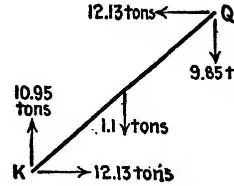

FIG 117.

The forces acting on the brace may now be found by the principle of action and reaction; they are represented on the free body diagram of this member (Fig. 117). As a check, the conditions $\Sigma F_x = \Sigma F_y = 0$ are applied and found to hold true.

71. Ropes and Pulleys. — The loads which act on a crane are usually applied through some sort of hoisting rig, generally a combination of pulleys and ropes or chains. We shall now devote particular attention

to the forces which such a rig exerts upon the parts to which it is connected. We shall assume throughout that the tensions T_1 and T_2 (Fig. 118) in the rope or chain on opposite sides of the pulley on which it bears are equal; this assumption implies perfect flexibility of the rope or chain and a frictionless pin supporting the pulley. Figure 118 is a free body diagram for the system comprising the pulley and the short portion of the rope shown; it is apparent that the reaction R of the pin against the pulley must bisect the angle between the equal forces T_1 and T_2 and is equal to $2\ T \cos \frac{1}{2}\alpha$. The actual force exerted by the rope against the pulley is distributed over that part of the circumference of the pulley against which the rope bears. It is, however, evidently equivalent to the forces T_1 and T_2 acting along the axes of the straight portions of the rope. If T_1 and T_2 are considered together they may be regarded as both applied at the points of tangency, or at the center of the pulley, regardless of rope thickness, since such displacements of *both* lines of action would not affect the resultant of T_1 and T_2.

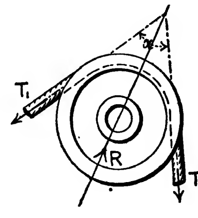

FIG. 118

A rope may be regarded either as a part of the crane, or as an external body that exerts forces on the crane at points of contact therewith. If the rope is regarded as part of the crane, we imagine it cut (as in the case of a truss member) at suitable sections; the tension at each such section then becomes an external force (pull) acting on the remaining portion of the rope. If the rope is regarded as a body separate from the crane, then the external forces which it exerts at every point where it makes contact with the crane must be considered.

EXAMPLE 1. Figure 119 represents a crane supported in a footstep bearing at the floor and a collar bearing at the wall bracket H. The hoisting rig consists of a simple

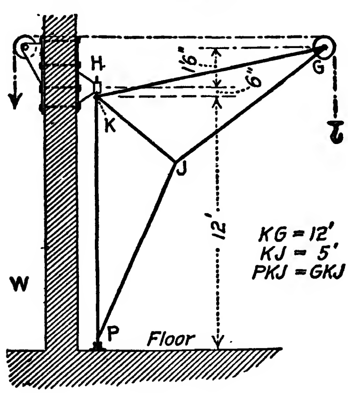

FIG. 119

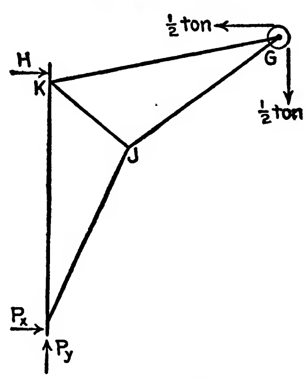

FIG. 120

hand winch mounted on the wall at W, a chain, and pulleys as shown. Pulley at G is 12 in. in diameter; the load is 0.5 tons; the weights of the members may be neglected. It is required to analyze the crane.

Solution: The free body diagram for the crane as a whole (chain removed) is shown in Fig. 120. The external forces are the pressure of the chain on the pulley (equivalent to the two components of 0.5 tons each); the horizontal reaction H of the wall bracket (sense assumed); and the reaction (direction unknown) at the footstep, represented by its components P_x and P_y. Solution of this system is effected as follows:

$$\Sigma M_P = 0 \text{ gives } H = 0.087 \text{ tons,}$$
$$\Sigma F_x = 0 \text{ gives } P_x = 0.413 \text{ tons,}$$
and
$$\Sigma F_y = 0 \text{ gives } P_y = 0.5 \text{ tons.}$$

All members except the vertical are simple two-force members, and the stresses in them can be found exactly as in the case of a truss. Solution is made first at joint G, where there are but two unknown stresses; then joint J can be considered. The results are:

$GK = 0.35$ tons tension; $GJ = 1$ ton compression; $JK = 0.57$ tons compression; $JP = 1$ ton compression.

The vertical member is subjected to the reactions H, P_x and P_y (previously determined) and, in addition, to the following forces exerted by the members named: A pull along GK equal to 0.35 tons; a push along JK equal to 0.57 tons; a push along JP equal to 1.0 ton.

EXAMPLE 2. Suppose the crane of Ex. 1 to be made with the hoisting chain carried to a winch on the post instead of through the wall to a separate hoisting device as before. It is required to analyze the crane.

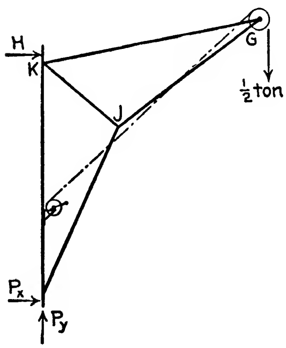

FIG. 121

Solution: The free body diagram for the entire crane (chain included) is shown in Fig. 121. The external forces are the load (applied to the hook); the horizontal reaction H of the wall bracket (sense assumed), and the reaction (direction unknown) at the footstep, represented by its components P_x and P_y. Solution of this system is effected as follows:

$$\Sigma M_P = 0 \text{ gives } H = -0.473 \text{ tons,}$$
$$\Sigma F_x = 0 \text{ gives } P_x = 0.473 \text{ tons,}$$
and
$$\Sigma F_y = 0 \text{ gives } P_y = 0.5 \text{ tons.}$$

The stresses in the two-force members are determined as in Ex. 1; their values are different because the chain pressure against the pulley at G is different. The forces acting on the vertical member are also determined as before, and consist of the reactions at H and P and the pushes and pulls of the attached members, GK, JK and JP.

(The student should draw the free body diagram for the crane discussed above with the chain removed, and should compare with Fig. 121 and also with Fig. 120. He should perceive, and prove to his own satisfaction, that if the hoisting chain is carried directly from the pulley at G to a winch mounted *anywhere on the crane*, the reactions at H and P will be exactly the same as in this example, whereas if the chain is carried to a winch *externally* mounted — as on the floor or wall — the reactions will depend upon the position of the winch, or, more precisely, on the direction of the hoisting cable leading thereto.

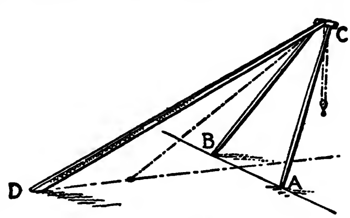

FIG. 122

EXAMPLE 3. Figure 122 represents a shear leg crane. It consists of two front legs AC and BC and a back leg CD, all connected by a horizontal pin at C; the front legs are pin-supported on the ground at A and B, and the back leg

is restrained at the ground by a holding-down rail and a long horizontal screw which works in a nut on the lower end D. The purpose of the screw is to move D, thus turning the front legs about AB and moving the load in and out. The following data are assumed: lengths of front legs 160 ft., distance between their lower ends 50 ft., distance between their upper ends 10 ft., length of back stay 210 ft., weight of each front leg 44 tons, weight of back leg 53 tons, weight of pin at C negligible. It is required to determine the pressure on the ends of the legs due to their own weights when the crane is in its position of greatest overhang (64 ft.).

Solution: First the entire crane is considered. The free body diagram is shown in Fig. 123; the external forces are the holding down force D_y exerted by the rail, the horizontal force D_x exerted by the screw, and the reactions at A and B, each represented by its x, y and z components, parallel to the assumed rectangular axes shown. The force system is noncoplanar, nonconcurrent, and nonparallel. For such a system (Art. 50) there are six independent conditions of equilibrium; application of these six conditions yields the following equations:

$$\Sigma F_x = A_x + B_x - D_x = 0 \quad (1)$$
$$\Sigma F_y = A_y + B_y - D_y - 53 - 44 - 44 = 0 \quad (2)$$
$$\Sigma F_z = -A_z + B_z = 0 \quad (3)$$
$$\Sigma M_x = -(A_y \times 25) + (B_y \times 25) + (44 \times 15) - (44 \times 15) = 0 \quad (4)$$
$$\Sigma M_y = (A_x \times 25) - (B_x \times 25) = 0 \quad (5)$$
$$\Sigma M_z = (D_y \times 87.6) + (53 \times 11.8) - (44 \times 32 \times 2) = 0 \quad (6)$$

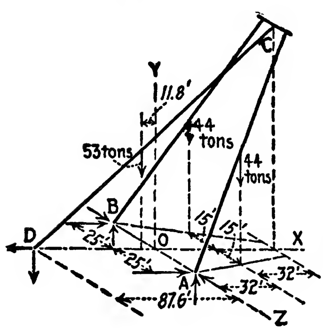

FIG. 123

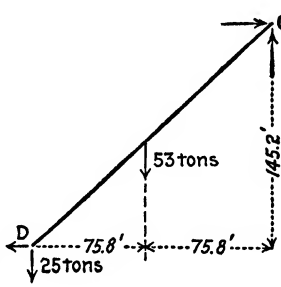

FIG. 124

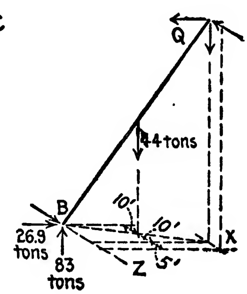

FIG. 125

Equation (6) shows that $D_y = 25$ tons; (4) shows that $A_y = B_y$; from these results and (2) it follows that A_y and B_y equal 83 tons. No other unknowns can be determined from the equations; but (3) shows that $A_z = B_z$, (5) that $A_x = B_x$, and (1) that $A_x + B_x = D_x$.

Next the back leg is considered. The free body diagram is shown in Fig. 124; the external forces are the weight, the forces D_x and D_y, and the pin pressure at C, represented by its components C_x and C_y. For this system

$$\Sigma M_C = (25 \times 151.6) - (D_x \times 145.2) + (53 \times 75.8) = 0, \text{ whence } D_x = 53.8 \text{ tons},$$
$$\Sigma F_x = C_x + 53.8 = 0, \text{ whence } C_x = 53.8 \text{ tons},$$
$$\text{and} \quad \Sigma F_y = C_y - 25 - 53 = 0, \text{ whence } C_y = 78 \text{ tons}.$$

Returning now to the equations that apply to the crane as a whole, it is found from equations (1) and (5) that

$$A_x = B_x = 26.9 \text{ tons}.$$

Next, one of the front legs (the farther) is considered. The free body diagram is shown in Fig. 125; the external forces are the weight, the pin pressure at the lower end (represented by rectangular components two of which have been determined),

and the pin pressure at the upper end, represented by its three rectangular components none of which is known. Solution is effected as follows:

ΣM (about vertical line through Q) $= (B_z \times 64) - (26.9 \times 20) = 0$, whence $B_z =$ 8.41 tons.

$\Sigma F_x = 26.9 - Q_x = 0$, whence $Q_x = 26.9$ tons.

$\Sigma F_y = 83 - 44 - Q_y = 0$, whence $Q_y = 39$ tons.

and $\Sigma F_z = 8.41 - Q_z = 0$, whence $Q_z = 8.41$ tons.

CHAPTER V

FRICTION

§ 1. Definitions and General Principles

72. Nature of Friction. — When one body slides or tends to slide over another, then the sliding of the first or its tendency to slide is resisted by the second. Thus, if A (Fig. 126) is a body which slides or tends to slide toward the right over B, then B is exerting some such force as R on A, and the component of R along the surface of contact is the resistance which B offers to the sliding or tendency. Of course A exerts on B a force equal and opposite to R; either of these equal forces is called the *total reaction* between the two bodies. The component of either total reaction along the (plane) surface of contact is called *friction*, and the component of either along the normal is called *normal pressure;* they will be denoted by F and N respectively. If the surface of contact of the two bodies is not plane, the force exerted at each elementary part of the surface is the total reaction at that element, and its components in and normal to the element are the friction and the normal pressure at the element. Friction is called *kinetic* or *static* according as sliding does or does not take place. Only static friction is considered here.

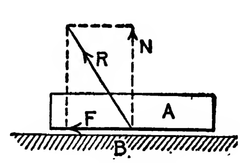

FIG. 126

The amount of static friction between two bodies depends upon the degree of the tendency to slip. Thus suppose that A (Fig. 127) is a block

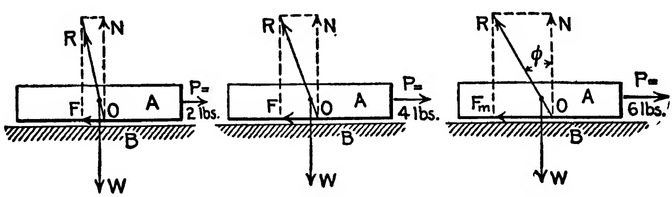

FIG. 127

weighing 10 pounds, upon a horizontal surface B; that the block is subjected to a horizontal pull P, and that the pull must exceed 6 pounds to start the block. Obviously when P = 2 pounds, say, then F = 2; when P = 4 pounds, then F = 4; etc., until motion begins. So long as P does not exceed 6 pounds, F equals P; that is, F is passive and changes just as P changes. And so in general we may say that friction is a passive force, which comes into action only to prevent a slipping that other forces tend to cause, and which is, in any given case, only so large as may be neces-

sary to prevent that slipping. It increases as the tendency to slip increases, and has its greatest value when slipping impends.

The inclination of the total reaction also depends upon the degree of the tendency to slip. Thus in the example above, when $P = 2$ pounds the angle $NOR = \tan^{-1} 2/10 = 11° 19'$; when $P = 4$ pounds, $NOR = \tan^{-1} 4/10 = 21° 48'$, etc., until motion begins, the greatest value obtaining when motion impends.

73. Coefficient of Friction. — The friction corresponding to impending motion is called *limiting friction.* We will denote it by F_m, since it is a maximum value (see Fig. 127).[1] The *coefficient of static friction* for two surfaces is the ratio of the limiting friction corresponding to any normal pressure between the surfaces and that normal pressure. We will denote it by μ; then

$$\mu = F_m/N, \quad \text{or} \quad F_m = \mu N; \text{ also, } F \ngtr \mu N.$$

The value of the coefficient of static friction between two surfaces depends upon several different factors, discussed in Art. 75 below.

74. Angle of Friction; Angle of Repose. — The *angle of friction* for two surfaces is the angle between the directions of the normal pressure and the total reaction when motion is impending. We will denote it by ϕ (see Fig. 127); then

$$\tan \phi = F_m/N; \text{ hence } \tan \phi = \mu.$$

If a block were placed upon an inclined plane, the inclination at which slipping would impend is called the *angle of repose* for the two rubbing surfaces; it will be denoted by ρ.

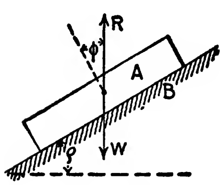

FIG. 128

The angles of friction and repose for two surfaces are equal; proof follows: Suppose that A (Fig. 128) is on the point of sliding down the incline; two forces act on A, its own weight W and the reaction R of the plane. Since A is at rest, R and W are colinear, that is, R is vertical; and since motion impends, the angle between R and the normal is the angle of friction ϕ. It follows, from the geometry of the figure, that ϕ and ρ are equal.

75. Experimental Determination of Coefficient of Friction. — The coefficient of static friction for two bodies A and B may be found in several ways: (i) Place A on B as in Fig. 127, and determine the pull P which will just start A; then $\mu = P$ divided by the weight of A. Or (ii) tilt B, and determine the inclination at which gravity will start A down; then μ equals the tangent of that angle of inclination. In either method several determinations must be made to obtain a fair average. Many experiments have been made in these ways, and it has been ascertained that coefficients of static friction depend on the nature of the materials, character of rubbing surfaces and kind of lubricant, if any be used. Early experimenters

reported (Coulomb 1871, Rennie 1828, Morin 1834, and others) that the coefficient is independent of the intensity of normal pressure; and although this announcement was clearly subject to the limitation of the range of the experiments performed, yet it was generalized and long accepted as a universal law of friction. But the universality of the law has been questioned; Morin himself pointed out that length of time of contact of the two bodies influences the coefficient; and obviously the coefficient changes when the intensities of pressure get so low that a considerable part of the friction is due to adhesion, or so high as to affect the character of the surfaces in contact. Messiter and Hanson report[1] practical constancy of coefficient for yellow pine and spruce. They give the following for planed or sandpapered (1) yellow pine and (2) spruce.

(1) $\mu = 0.25$ to 0.32; average $\mu = 0.29$ for 100 to 1000 lbs. per sq. in.
(2) $\mu = 0.18$ to 0.53; average $\mu = 0.42$ for 100 to 1600 lbs. per sq. in.

The variation depends on relation of grain of wood to direction of slide.

Coefficients of Static Friction

(Compiled by Rankine from experiments by Morin and others.)

Dry masonry and brickwork	0.6 to 0.7
Masonry and brickwork with damp mortar	0.74
Timber on stone	about 0.4
Iron on stone	0.3 to 0.7
Timber on timber	0.2 to 0.5
Timber on metals	0.2 to 0.6
Metals on metals	0.15 to 0.25
Masonry on dry clay	0.51
Masonry on moist clay	0.33
Earth on earth	0.25 to 1.0
Earth on earth, dry sand, clay, and mixed earth	0.38 to 0.75
Earth on earth, damp clay	1.0
Earth on earth, wet clay	0.31
Earth on earth, shingle and gravel	0.81 to 1.11

§ 2. General Applications

76. Test for Rest or Motion; Algebraic Solution. — A common problem involving friction may be stated as follows: A body is supported so that it can slip and is subjected to given forces; it is required to ascertain whether those forces do cause slipping, and to determine the value of the friction.

Solution can often be effected as follows: Assume that the body is at rest and draw the free body diagram. At places of contact where motion would involve sliding, represent the force exerted on the body by its com-

[1] *Eng. News*, 1895, Vol. 33, page 322.

ponents, — the friction F and the normal pressure N. Solve for these by applying the appropriate conditions of equilibrium. Then compare F with μN. If F is less than μN there is no motion, and the computed value of F is correct; if F is greater than μN there is motion, and the computed value of F is incorrect, the actual friction being kinetic and less than μN. This method is illustrated in the examples below.

Sometimes a body in equilibrium (or so assumed) under circumstances that involve friction will be acted upon by a force system that includes so many unknowns as not to be solvable. It is then impossible to test for rest or motion by the method just explained. It is, however, often possible to ascertain whether or not equilibrium exists by determining (according to the methods described in the next article) what " applied " or " active " forces would be required to cause slipping under the given conditions of support. If the actual applied forces are less than the required forces, or are so located as to have less disturbing effect, then it is safe to conclude that equilibrium exists, though the supporting forces have not been actually determined. This method is illustrated by the examples of Art. 77.

EXAMPLE 1. A block rests upon a horizontal floor; the block weighs 150 lbs. and the coefficient of friction between it and the floor is 0.3. A pull of 40 lbs., acting upward at an angle of 30° to the horizontal, is applied to the block. It is required to determine whether or not the block slips.

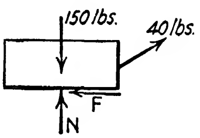

FIG. 129

Solution: The block is assumed to be in equilibrium and the free body diagram is drawn (Fig. 129). The forces acting on the block are its own weight, the pull of 40 lbs. and the reaction of the floor, which consists of the normal component N, acting up, and the friction component F, acting to the left to oppose motion.[1] The equation

$$\Sigma F_y = -150 + 40 \sin 30° + N = 0$$

gives $N = 130$ lbs., whence $\mu N = 39$ lbs. The equation

$$\Sigma F_x = -F + 40 \cos 30° = 0$$

gives $F = 34.64$ lbs. On comparison, it is seen that F is less than μN (that is, the friction required to maintain equilibrium is less than the limiting friction); therefore there is no slipping.

[1] Here, and subsequently in examples in which the body discussed is a block or similar object whose dimensions are not specified, no especial attempt will be made to represent the lines of action of the forces in their correct relative *position*. In the above example the three forces — the weight of the block, the applied pull and the reaction of the floor — must be concurrent, and consideration of the free-body diagram will show that the normal component of the reaction must therefore be somewhere to the left of the line of action of the weight instead of colinear with it as shown. But since solution is effected by the conditions $\Sigma F_x = \Sigma F_y = 0$, the exact position of N is immaterial.

It is apparent that the applied pull might have such a magnitude and line of action as to cause the block to tip or overturn, but it will be assumed in all examples of this type that this does not occur. This assumption renders the use of moment equations unnecessary, and so the positions of the lines of action of the forces need not be further considered.

EXAMPLE 2. A uniform ladder of length l is placed with its upper end against a smooth vertical wall and its lower end on a horizontal rough floor; the inclination of the ladder to the horizontal is 55°; the coefficient of friction between floor and ladder is 0.5. It is required to determine whether the ladder will remain in this position or slip down.

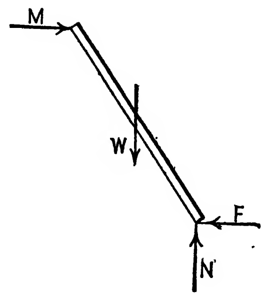

FIG. 130

Solution: The ladder is assumed to be in equilibrium and the free body diagram is drawn (Fig. 130). The forces acting on the ladder are its own weight W, the horizontal reaction of the smooth wall M, and the reaction of the floor, which consists of the normal component N acting up and the friction component F acting to the left to oppose motion.

The equation

$$\Sigma F_y = -W + N = 0$$

gives $N = W$, whence $\mu N = .5\,W$. The equation

$$\Sigma F_x = M - F = 0$$

gives $M = F$. The external force system consists therefore of two couples, one made up of W and N, the other of F and M. The sum of the moments of these couples must equal zero, or

$$\Sigma M = (W \times l/2 \cos 55°) - (F \times l \sin 55°) = 0, \text{ whence } F = 0.35\,W.$$

On comparison it is seen that F is less than μN, therefore there is no slipping.

EXAMPLE 3. A block weighing 120 lbs. rests on a platform that is inclined at 20° to the horizontal; the coefficient of friction between the block and the plane is 0.25. A horizontal force of 30 lbs., parallel to the plane of the platform, is applied to the block as shown in Fig. 131. It is required to determine whether or not the block slips.

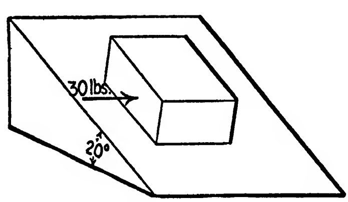

FIG. 131

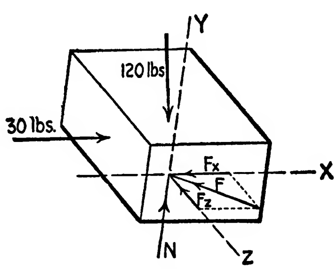

FIG. 132

Solution: The block is assumed to be in equilibrium and the free body diagram is drawn (Fig. 132). Rectangular axes are assumed, the x-axis parallel to the 30 lb. force, the y-axis normal to the plane, and the z-axis perpendicular to these. The external forces acting on the block are its weight W, the 30 lb. force, and the reaction from the plane. This reaction is represented by its normal component N and its friction component F, the latter in turn being represented by its rectangular components F_x and F_z. The external system is noncoplanar, and appropriate conditions of equilibrium are: $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.

Application of these conditions yields the following equations:

$$\Sigma F_y = -120 \cos 20° + N = 0, \text{ whence } N = 112.7 \text{ lbs. and } \mu N = 28.2 \text{ lbs.}$$
$$\Sigma F_x = 30 - F_x = 0, \text{ whence } F_x = 30 \text{ lbs.}$$
$$\Sigma F_z = -120 \sin 20° + F_z = 0, \text{ whence } F_z = 41 \text{ lbs.}$$

F, the total friction, is the resultant of F_x and F_z; hence $F = (30^2 + 41^2)^{\frac{1}{2}} = 50.8$ lbs.

On comparison it is seen that F is greater than μN, therefore the block will slip.

77. Test for Rest or Motion; Graphical Solution. — As before, we assume that the body is at rest and draw the free body diagram, but instead of representing the forces at places of contact by their friction and normal components we consider, at each such place, the total force, and ascertain whether or not equilibrium is possible by use of the so-called *cone of friction*, which may be described as follows: The force exerted on any body A (Fig. 133) by another body B (not shown) which bears against it at any point O[1] cannot be inclined at a greater angle to the normal NO than ϕ, the friction angle for the surfaces in contact. Therefore R represents this force in one limiting position. But the force can assume this limiting angle ϕ to the normal while acting in any vertical plane through O, and so its line of action may lie anywhere inside the cone COC' generated by revolving R about the normal ON. This cone is the cone of friction for the surfaces in contact at O.

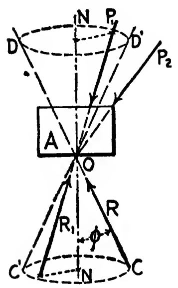

FIG. 133

If A be thought of as resting on or against B, and being supported thereby, then any force that can be balanced by a reaction R lying within the cone of friction may be applied to A without causing slipping. Thus any force P_1, passing through O and lying within the cone COC' (or, what amounts to the same thing, DOD') would be balanced by an equal, opposite and colinear reaction R_1. But any force P_2 lying without the cone DOD' would not be so balanced, because no possible reaction would be colinear with it. The point O can of course be anywhere in the contact surface common to A and B; thus if B is a floor on which A rests, the apex of the cone of friction can be anywhere in the bottom of A.

When there are a number of different places at which motion would involve sliding, the several corresponding cones of friction indicate all possible positions of the supporting forces there exerted, and make it possible to determine whether or not such supporting forces will prevent slipping under the given circumstances.

EXAMPLE 1. A block weighing 120 lbs. is placed on a plane inclined at 40° to the horizontal; the coefficient of friction between the block and the plane is 0.2. A horizontal force of 50 lbs. is applied to the block, so as to tend to cause slipping up the plane. It is required to determine whether or not the block slips.

[1] The force exerted by B is here assumed to be concentrated. If the force is distributed, then O represents the center of pressure, and for present purposes the force may be regarded as concentrated at this point. It is to be noted that O can be anywhere in the surface of contact — thus in Fig. 133 O, the apex of the cone of friction, might be taken anywhere on the lower face of A.

Solution: The line of action of the resultant R of the 50- and 120-pound forces is determined by the parallelogram of forces (Fig. 134). At the point O where R cuts the bottom of the block the cone of friction is constructed, the friction angle $\phi = \tan^{-1} 0.2$ being laid off to either side of the normal ON. It is seen that R lies outside the cone; therefore the reaction of the plane cannot assume a position colinear with it; therefore the block will slip down.

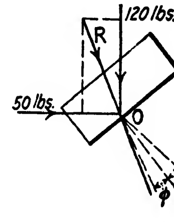

FIG. 134

EXAMPLE 2. A uniform ladder is placed with its upper end against a smooth vertical wall and its lower end on a rough horizontal floor. The inclination of the ladder to the horizontal is 55° and the coefficient of friction between the ladder and the floor is 0.5. It is required to determine whether the ladder will remain in this position or slip down. (Same as Ex. 2, Art. 76.)

Solution: The ladder is assumed to be in equilibrium and the free body diagram constructed (Fig. 135). There are three external forces, the weight W, the horizontal reaction of the smooth wall M, and the reaction of the floor R. If these three forces are in equilibrium they must be concurrent, and R must pass through 1, the intersection of M and R, as represented. At the lower end of the ladder the cone of friction is constructed, the friction angle $\phi = \tan^{-1} 0.5$ being laid off to either side of the normal ON.[1] It is seen that R falls inside the cone of friction, therefore it represents a reaction which the floor is able to exert, and so the ladder will not slip.

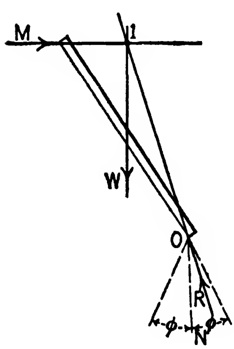

FIG. 135

78. Equilibrium when Slipping Impends.

— A body is supported so that it can slip; it is required to determine something as to the forces which, applied to the body, would cause slipping in some stated direction to impend.

By the conditions of the problem the body is known to be in equilibrium. We draw the free body diagram and in representing the external forces at contact points make use of one or the other of the facts that, since slipping impends: (i) The friction component is equal to the product of the coefficient of friction and the normal pressure, and opposes the impending slipping; and (ii) The total force is inclined at the friction angle ϕ to the normal, and acts so as to oppose the impending slipping. We then apply the appropriate conditions of equilibrium and solve for the unknown quantities to be determined.

If the solution is to be effected by algebraic methods we may deal either with the total force at each point of contact, or with its friction and normal components. If graphical methods are to be employed, it is usually advantageous to deal with the total force.

EXAMPLE 1. A block of weight W rests on a horizontal floor; the coefficient of friction between the block and the floor is μ. It is required to determine the magnitude of

[1] Frequently, as here, it is evident which side of the normal the reaction must be on, and when this is the case only one element of the cone — that on the same side of the normal as the reaction — is really required.

the force P, inclined at an angle θ to the horizontal, which must be applied to the block to cause it to slip.

Solution: The free body diagram of the block is shown in Fig. 136. The external forces are the weight W, the applied force P, and the reaction of the floor, represented by its normal component N and its friction component $F_m = \mu N$ (limiting friction because motion impends). The following equations apply:

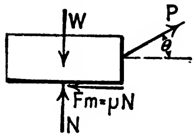

FIG. 136

$$\Sigma F_y = -W + P\sin\theta + N = 0,$$

and

$$\Sigma F_x = P\cos\theta - \mu N = 0.$$

Solution of the first equation for N gives $N = W - P\sin\theta$. Substitution of this value of N in the second equation gives $P\cos\theta - \mu(W - P\sin\theta) = 0$. Solution of this equation for P gives $P = \dfrac{\mu W}{\cos\theta + \mu\sin\theta}$ which, on substitution of $\tan\phi$ for μ, gives $P = \dfrac{W\sin\phi}{\cos(\theta - \phi)}$.

From this equation it is easy to show that P has its minimum value when $\theta = \phi$ and is then equal to $W\sin\phi$; that when $\theta = 0$, $P = W\tan\phi$; that when $\theta = -(90 - \phi)$, P is infinite (that is, no force, however great, applied at such an angle, will cause the block to slip to the right); and that when $\theta < -(90 - \phi)$, P becomes negative (showing again that no force, applied at such an angle, will cause slipping to the right).

EXAMPLE 2. A block weighing 200 lbs. rests on a plane inclined at an angle of 30° to the horizontal; the coefficient of friction between the block and plane is 0.8. It is required to determine what horizontal force will (*a*) make the block slip up the plane; (*b*) make the block slip down the plane.

Solution: (*a*) The free body diagram for the block is shown in Fig. 137. The external forces are the weight W, the applied force P, and the reaction of the plane represented by its normal component N and its friction component F_m (acting *down* the plane to oppose the impending upward slipping). Assuming x- and y-axes respectively parallel and normal to the plane and noting that $F_m = 0.8\,N$, the following equations may be written:

$$\Sigma F_x = P\cos 30° - 200\sin 30° - 0.8\,N = 0,$$

and

$$\Sigma F_y = -200\cos 30° - P\sin 30° + N = 0.$$

Simultaneous solution of these equations gives $P = 512$ lbs.

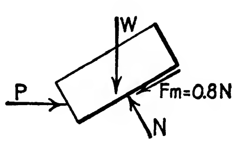

FIG. 137

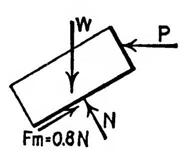

FIG. 138

(*b*) The free body diagram for the block when motion impends down the plane is shown in Fig. 138. The external forces are W, P, N and F_m (acting *up* the plane to oppose the impending downward slipping). Assuming axes as before, the following equations apply:

$$\Sigma F_x = -P\cos 30° - 200\sin 30° + 0.8\,N = 0,$$

and

$$\Sigma F_y = -200\cos 30° + P\sin 30° + N = 0.$$

Simultaneous solution of these equations gives $P = 30.5$ lbs.

(For certain values of the coefficient of friction and slope of plane, a negative value of P will be obtained in either of the above cases. The student should ascertain, for each case, the significance of this negative result.)

EXAMPLE 3. A block A rests on the horizontal top of a second block B, which in turn rests on a horizontal floor C (Fig. 139). A is attached to the floor by an inclined cord as shown. A weighs 60 lbs.; B weighs 140 lbs.; the coefficient of friction between A and B is 0.3; the coefficient of friction between B and C is 0.2. It is required to determine the magnitude of the horizontal force P which, applied to B as shown, will cause it to slip from under A.

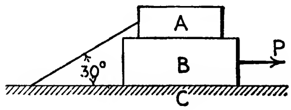

FIG. 139

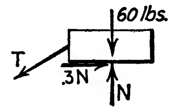

FIG. 140

Solution: Block A is considered first; the free body diagram is shown in Fig. 140. The external forces are the weight, the pull of the cord T, and the reaction from B, represented by its normal component N and its friction component $0.3\,N$ (the sense of the friction is to the *right*, because B, in slipping out from under A, tends to drag A along by friction, and so exerts this friction towards the right). The following equations may be written for this system:

$$\Sigma F_x = -T\cos 30° + 0.3\,N = 0,$$
$$\Sigma F_y = -60 - T\sin 30° + N = 0.$$

Simultaneous solution of these equations for N (T is not required) gives

$$N = 72.6 \text{ lbs.}; \quad \text{whence} \quad 0.3\,N = 21.8 \text{ lbs.}$$

Next the block B is considered; the free body diagram is shown in Fig. 141. The external forces are the weight, the applied force P, the forces exerted by block A (equal and opposite to N and $0.3\,N$ found above), and the reaction of the floor, represented by its normal component M and its friction component $0.2\,M$ (the friction acts to the left to oppose the impending slipping). The following equations apply:

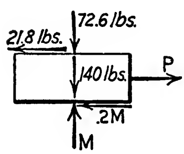

FIG. 141

$$\Sigma F_x = P - 21.8 - 0.2\,M = 0,$$
and
$$\Sigma F_y = -72.6 - 140 + M = 0.$$

Solution of these equations for P (M is not required) gives $P = 64.3$ lbs.

EXAMPLE 4. A uniform ladder of length l is to be placed with its upper end against a smooth vertical wall and its lower end on a rough horizontal floor, the coefficient of friction between the ladder and the floor being known. It is required to determine the least angle to the horizontal at which the ladder can be thus placed without slipping down under its own weight.

Solution: The free body diagram for the ladder, assumed on the point of slipping down, is shown in Fig. 142. The external forces are the weight, the horizontal reaction M of the smooth wall, and the reaction of the floor, represented by its normal component N and its friction component; the latter has its limiting value μN and acts to the right to oppose the slipping that impends. The following equations apply:

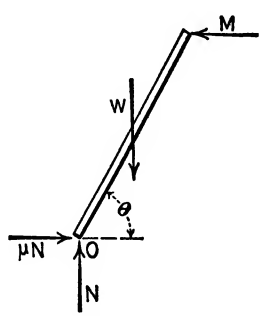

FIG. 142

$$\Sigma F_y = -W + N = 0,$$
$$\Sigma F_x = \mu N - M = 0,$$
$$\Sigma M_o = (M \times l\sin\theta) - (W \times \tfrac{1}{2}l\cos\theta) = 0.$$

The first equation shows $N = W$, therefore $\mu N = \mu W$; and (from the second equation) $M = \mu W$. Substitution of this value of M in the third equation and solving gives

$$\theta = \cot^{-1} 2\,\mu.$$

EXAMPLE 5. Figure 143 represents a type of simple hanger and the vertical rod which supports it. The hanger consists of a forked arm with a hole in each part of the fork, permitting it to be slipped over the rod as shown. If properly constructed, the hanger will not slip down the rod on account of its own weight or that of a load unless it be applied close to the fork. It is required to determine how far in — that is, how close to the rod — a load may be applied without causing the hanger to slip down. It is assumed that the coefficient of friction between the hanger and the rod is known, and that the weight of the hanger is negligible in comparison with the load to be supported.

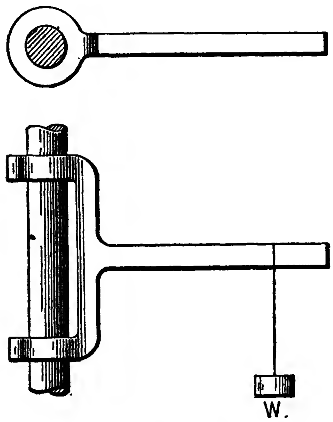

FIG. 143

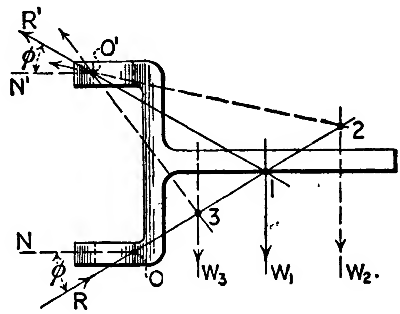

FIG. 144

Solution: The free body diagram for the hanger is shown in Fig. 144. There are three external forces, the reaction R of the rod against the lower arm of the hanger at O, the reaction R' against the upper arm at O', and the load W. When slipping impends R and R' act as shown, their lines of action being inclined to the normals ON and $O'N'$ an amount equal to the angle of friction ϕ. Since R, R' and W are in equilibrium, they must be concurrent as shown; therefore to just put the hanger on the point of slipping the load must be hung from a point on the vertical through 1. W_1, then, represents the load in the position which would correspond to impending slipping.

If the load is hung farther out, as indicated by W_2, the hanger will not slip, because the reactions at O and O' can intersect at some point on the line of action of W_2, as 2, without falling outside of the cones of friction at O and O' respectively.[1] If the load is hung farther in, as indicated by W_3, the hanger will slip, because the reactions R and R' cannot intersect on the line of action of W_3 while remaining inside the cones of friction at O and O'.

EXAMPLE 6. A beam is placed horizontally across a trough formed by two inclined planes (Fig. 145). The coefficients of friction between the planes and the beam, and the inclinations of the planes are given. It is required to determine the limiting positions (if any) between which a load can be placed on the beam without causing it to slip. It is assumed that the weight of the beam is negligible in comparison with the weight of the load to be placed on it.

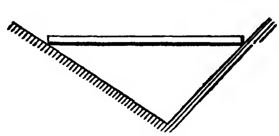

FIG. 145

Solution: The free body diagram for the beam is shown in Fig. 146. There are three external forces, the load W, the reaction R at the left end and the reaction R' at the right end. When slipping to the right impends, the reactions will as-

[1] It should be noted that in this case the actual values and directions of R and R' are indeterminate. The indicated lines of action of these forces merely represent one possible condition.

sume the positions indicated by R_1 and R_1'. Since R_1, R_1' and W are in equilibrium, they must be concurrent; therefore to just put the beam on the point of slipping to the right the load must be applied at a point on the vertical through 1. W_1 represents a load so applied. If the load is moved farther to the left, the beam will slip to the right, because it will no longer be possible for the reactions to intersect on the line of action of W; if the load is moved somewhat farther to the right, the beam will not slip because it will be possible for the reactions to so intersect.

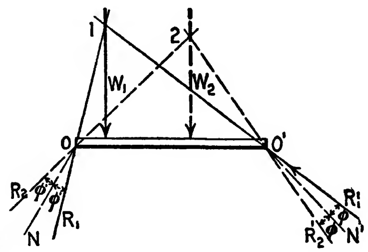

FIG. 146

In the same way it can be shown that W_2 represents the load in the position where it will just put the beam on the point of slipping to the left. If the load is moved farther to the right the beam will slip to the left; if the load is moved somewhat farther to the left the beam will not slip. The load may therefore be applied anywhere between W_1 and W_2 without causing the beam to slip in either direction.

§ 3. Friction in Some Mechanical Appliances

79. The Wedge. — Figure 147 illustrates a typical application of the wedge; its use makes possible the overcoming of a large load W by means of a relatively small applied force P. In order that P may start the wedge inward and thus overcome the load W, the friction at the three rubbing surfaces must be overcome also. To determine the necessary value of P, the block M and the wedge are separately considered as bodies in equilibrium with slipping impending, and the several forces involved solved for by the methods that have been discussed. If the three rubbing contacts are equally rough and ϕ is their common angle of friction, then the free body diagram and force polygon for M are as shown in Fig. 148, and for

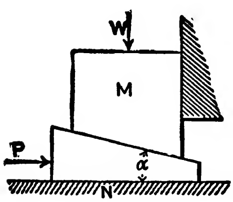

FIG. 147

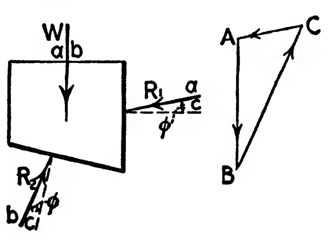

FIG. 148

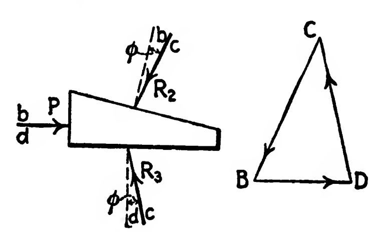

FIG. 149

the wedge as shown in Fig. 149 (the weights of M and the wedge are neglected). Solution for P is easily effected by either graphical or algebraic methods, the sine proportion relation (Art. 51) being especially convenient if algebraic methods are employed.

If it is desired to ascertain the force required to *withdraw* the wedge, it is necessary to assume M guided on the left. The free body diagram and force polygon for this condition (slipping to the left impending) are, for M, as shown in Fig. 150. For the wedge they are as shown in Fig. 151.

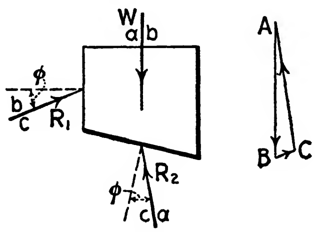

FIG. 150

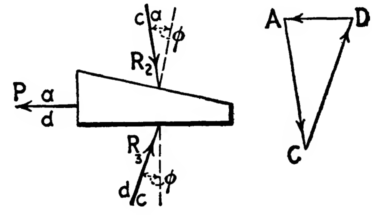

FIG. 151

$ABCA$ is the force triangle for the block M; $AB(W)$ and all the angles are known. Solving for $R_2(CA)$

$$\frac{R_2}{W} = \frac{\cos \phi}{\cos \alpha}, \qquad R_2 = \frac{\cos \phi}{\cos \alpha} W.$$

$ACDA$ is the force triangle for the wedge; $AC(R_2)$ and all the angles are known. Solving for $P(DA)$

$$\frac{P}{R_2} = \frac{\sin (2\,\phi - \alpha)}{\cos \phi},$$

$$P = \left(\frac{\sin (2\,\phi - \alpha)}{\cos \phi}\right) R_2 = \left(\frac{\sin (2\,\phi - \alpha)}{\cos \phi}\right)\left(\frac{\cos \phi}{\cos \alpha}\right) W = \frac{\sin (2\,\phi - \alpha)}{\cos \alpha} W.$$

Obviously if α is equal to $2\,\phi$, P is zero — that is, the wedge would be just on the point of slipping out as the result of the pressure from M. If α is greater than $2\,\phi$, the expression gives a negative value for P, which means that the wedge would slip out unless prevented by a force acting to the right (M in this case being guided on the right). If α is less than $2\,\phi$ the value of P is positive, which shows that the wedge would not slip out unless pulled. When this is the case the mechanism is said to be *self-locking*.

80. The Screw. — Figure 152 represents a simple jackscrew much used for raising and lowering heavy loads through short distances. In the simpler forms, the screw is turned by means of a lever stuck through a hole in the head H of the screw. There is frictional resistance between the screw and the nut, also between the cap C and the head of the screw, unless the load can turn with the screw. The relations between the load on the screw and the moment required to turn it so as to (1) raise and (2) lower the load, will now be determined.

Let Pa be the horizontal couple applied to the lever to turn the screw; W = load on the cap; r = mean radius of the screw, $\frac{1}{2}(r_1 + r_2)$; α = pitch

angle $= \tan^{-1}(h \div 2\pi r)$, where h = pitch; and $\phi = \tan^{-1}\mu$, where μ = coefficient of friction. If the friction between the cap and head of screw be disregarded, the screw can be considered as a body in equilibrium under the action of the couple Pa exerted by the lever and the reaction of the nut against the threads of the screw. At each point of contact between the screw and nut, the latter exerts a pressure dR whose normal and tangential component we call dN and dF respectively.

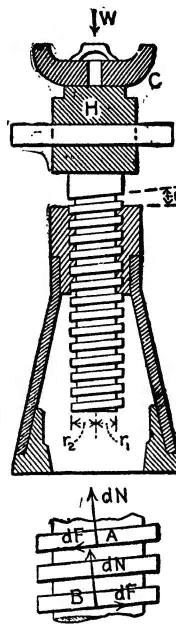

FIG. 152

(1) When the screw tends to rise, dF acts downward on the screw as shown at A; and when motion impends, the angle between dR and the vertical is $\phi + \alpha$. Taking the sum of the vertical components of all the forces acting on the screw, and the sum of the moments of all the forces about the axis of the screw, we get

$$-W + \Sigma\, dR \cos(\phi + \alpha) = 0,$$

or $$\cos(\phi + \alpha)\, \Sigma\, dR = W,$$

and $$P_1 a - \Sigma\, dR \sin(\phi + \alpha)\, r = 0,$$

or $$r \sin(\phi + \alpha)\, \Sigma\, dR = P_1 a,$$

whence $$P_1 a = \tan(\phi + \alpha)\, Wr$$

where $P_1 a$ = the moment required to *raise* the load.

(2) When the screw tends to descend, dF acts upward as shown at B; and when motion impends, the angle between dR and the vertical is $\phi - \alpha$. Taking the sum of the vertical components, and the sum of the moments as above, we get

$$-W + \Sigma\, dR \cos(\phi - \alpha) = 0, \quad \text{or} \quad \cos(\phi - \alpha)\, \Sigma\, dR = W$$

and $$P_2 a - \Sigma\, dR \sin(\phi - \alpha)\, r = 0, \quad \text{or} \quad r \sin(\phi - \alpha)\, \Sigma\, dR = P_2 a,$$

whence $$P_2 a = \tan(\phi - \alpha)\, Wr$$

where $P_2 a$ = the moment required to *lower* the load.

Obviously, if α is equal to ϕ, the moment required to lower the load is zero, which means that the screw would be just on the point of turning under the action of W alone. Of α is greater than ϕ, the expression gives a negative value for $P_2 a$, which means that a moment would have to be applied so as to *prevent* the screw from turning. If α is less than ϕ, the value of $P_2 a$ is positive, which means that the screw would not turn and allow the load to descend unless a moment of the indicated value were applied — that is, the mechanism is self-locking. Jackscrews are always made self-locking, the pitch angle being between 4 and 6 degrees generally,

while the coefficient of friction for ordinary conditions of lubrication is not less than about 0.15.

To allow for the friction between the cap and the head of the screw, let μ = the coefficient of friction, and R = the effective arm of the friction there with respect to the axis of the screw. (If the surface of contact between the cap and the head were flat and a full circle, R would equal two-thirds the radius of the circle. But the contact is generally a hollow circle, as in Fig. 152, and then R is practically equal to the mean radius.) The friction moment at the cap is μWR; therefore

(1) for raising the load, $Pa = Wr \tan (\phi + \alpha) + \mu WR,$

(2) for lowering the load, $Pa = Wr \tan (\phi - \alpha) + \mu WR.$

81. Journal in Worn Bearing. — Figure 153 represents, in section, a journal in a worn bearing, wear much exaggerated; the contact between the two is along a line practically. When the journal is about to turn clockwise and slip, then the bearing exerts a reaction R', making an angle ϕ (the angle of friction for the surfaces in contact) with the normal ON; when the journal is about to turn counterclockwise and slip, then the bearing exerts a reaction R'' inclined at an angle ϕ with ON, but on the other side. If the radius of the journal is r, then the perpendicular from the center to R' and R'' equals $r \sin \phi$, and the circle of radius $r \sin \phi$ with center at the center of the cross section of the journal is tangent to R' and R''. This circle is called the *friction circle* for journal and bearing. For nearly smooth contacts $\sin \phi$ nearly equals $\tan \phi$ or μ, and hence the radius of the circle practically equals μr.

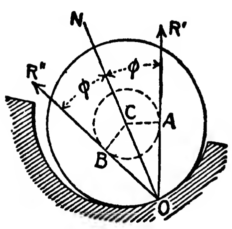

FIG. 153

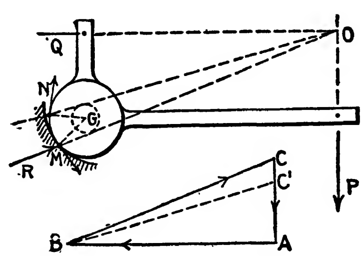

FIG. 154

We use the friction circle as an aid to fix upon the line of action of the reaction between journal and bearing when motion impends; the line is tangent to the circle. For example, consider the bell crank shown in Fig. 154, the journal being $1\frac{1}{4}$ inches in diameter and the coefficient of friction 0.3; the requirement is to determine the least force P, acting as shown, which will overcome Q (that is, start the bell crank to turn clockwise), and the pressure on the bearing then. The radius of the friction circle is $\frac{5}{8} \sin \tan^{-1} 0.3 = 0.18$ inch. Since there are but three forces acting on the bell crank (P, Q and R), they are concurrent, that is, R acts through

O; but R is also tangent to the circle as shown, and so its line of action is known. To determine the values of P and R, draw AB to represent Q by some scale, and lines through A and B parallel to P and R to their intersection C; then BC and CA represent the magnitudes and directions of R and P respectively.

(Which one of the two tangent lines to take can be determined by trial. Thus, trying ON, the contact between journal and bearing would be at N, and the tangential or frictional component of the pressure on the journal would be as shown, not consistent with the assumed tendency to slipping. Obviously the other tangent is the correct one, and on investigating for the friction component of R when acting at M it is found that such component is consistent with the assumed tendency to slip.)

The force P which would just permit Q to start the bell crank to turn counterclockwise could be determined in a similar way. Then R would act along the tangent ON, and P would be represented by $C'A$. When P has any value between $C'A$ and CA, then slipping does not impend, and the line of action of R cuts the friction circle.

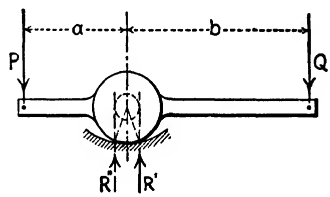

FIG. 155

As another example, consider a straight lever, supported and loaded as shown in Fig. 155. The reaction of the bearing is necessarily parallel to the vertical forces P and Q. If clockwise rotation impends the reaction is as represented by R', and

$$P(a + r \sin \phi) - Q(b - r \sin \phi) = 0, \text{ or } P = \frac{b - r \sin \phi}{a + r \sin \phi} Q.$$

If counterclockwise rotation impends, the reaction is as represented by R'' and

$$P(a - r \sin \phi) - Q(b + r \sin \phi) = 0, \text{ or } P = \frac{b + r \sin \phi}{a - r \sin \phi} Q.$$

For any given value of Q, equilibrium will obtain for any value of P between these extremes. If the lever shown is thought of as a scale beam, by means of which a load P is measured against a known counterweight Q, it is apparent that for the greatest sensitiveness r should be as small as possible, and so in such devices knife-edge bearings are employed. The conditions as to contact are then somewhat different from those assumed in the above discussion, but the relations between P and Q are similar.

82. Belt or Coil Friction. — Figure 156 represents a cylinder about a part of which a belt or rope is wrapped. If the cylinder is not very smooth, then the pulls P_1 and P_2 may be quite unequal without causing slipping over the cylinder, as may be easily verified by trial. When slipping impends, then the ratio of these pulls depends on the coefficient of friction

and on the angle of wrap. An expression for this ratio will now be developed.

Let P_1 be the smaller pull and P_2 be the larger pull when slipping impends, μ the coefficient of friction, and α the angle of lap expressed in radians. The forces acting upon the part of the belt in contact with the cylinder consist of the tensions P_1 and P_2, the normal pressure, and the friction (Fig. 157). Let p denote the normal pressure per unit length of arc; then the normal pressure on any part whose length is ds (enlarged in Fig. 158) is pds. The friction on that part may be called dF, and the ten-

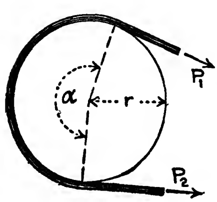

FIG. 156

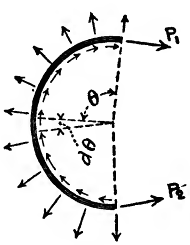

FIG. 157

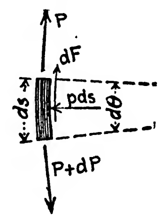

FIG. 158

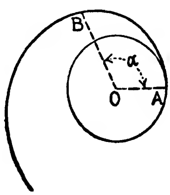

FIG. 159

sions P and $P + dP$. Since the part is at rest, $pds = 2\,P \sin \frac{1}{2}\,d\theta = Pd\theta$, or $p = P/r$; that is, the normal pressure per unit length at any point of the contact equals the belt tension there divided by the radius of the cylinder. When slipping impends, $dF = \mu pds$, and since $dF = dP$,

$$dP = \mu \frac{P}{r}\,ds, \text{ or } \frac{dP}{P} = \mu \frac{ds}{r} = \mu d\theta.$$

Integration gives $$\Big[\log_e P\Big]_{P_1}^{P_2} = \mu\Big[\theta\Big]_0^{\alpha};$$ hence,

$$\log_e P_2 - \log_e P_1 = \mu\alpha,$$ and

$$P_2 = P_1 e^{\mu\alpha} \text{ or } P_2/P_1 = e^{\mu\alpha}.$$

For a given value of P_1, P_2 increases very rapidly with α as shown by Fig. 159, which is the polar graph of the foregoing equation, P_2 and α being the variables, $e = 2.718$, μ taken as $\frac{1}{4}$, and $P_1 = O\,4$. The following table gives values of the ratio P_2/P_1 for three values of the coefficient of friction and for twelve values of the angle of lap.

MAXIMUM RATIOS P_2/P_1 (Slipping Impending)

$\frac{\alpha}{2\pi}$	μ			$\frac{\alpha}{2\pi}$	μ		
	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
0.1	1.17	1.23	1.37	0.7	3.00	4.33	9.00
0.2	1.37	1.51	1.87	0.8	3.51	5.34	12.34
0.3	1.60	1.87	2.57	0.9	4.11	6.58	16.90
0.4	1.87	2.31	3.51	1.0	4.81	8.12	23.14
0.5	2.19	2.85	4.81	2.0	23.	66.	535.
0.6	2.57	3.51	6.59	3.0	111.	535.	12,390.

EXAMPLE. Figure 160 represents a simple type of band brake. It consists of a rope or other band wrapped part way around a brake wheel W, the two ends of the band being fastened to the brake lever L; the lever is pivoted at Q. Obviously any force as P tightens the band, and if the wheel tends to turn (on account of some turning force, not shown), then P induces friction between wheel and band. It is required to determine how great a frictional moment (origin in the axis of the wheel) the force P can induce.

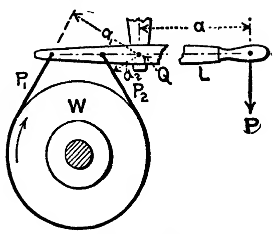

FIG. 160

Solution: Let M = the moment, P_2 = the larger tension in the brake band (on the side as marked when the wheel tends to rotate as indicated), P_1 = the smaller tension, r = radius of the wheel, a_1 = arm of P_1 with respect to Q, a_2 = arm of P_2, and a = arm of P. Consideration of the forces acting on the brake-strap shows that $M = (P_2 - P_1)r$; consideration of forces acting on the lever shows that $Pa = P_1a_1 + P_2a_2$. For a given P, M is greatest when slipping impends, and then $P_2 \div P_1 = e^{\mu\alpha}$. These three equations solved simultaneously show that

$$M = Pa(e^{\mu\alpha} - 1)r \div (a_2e^{\mu\alpha} + a_1).$$

For example, let P = 75 lbs., a = 10 ft., $\mu = \frac{1}{4}$, $\alpha = 320°$ (= 5.5 radians), r = 3 ft., a_1 = 2 ft., and a_2 = 9 inches. Then $\alpha \div 2\pi$ = about 9, and $e^{\mu\alpha}$ = 4.115 (see table on preceding page); and

$$M = 75 \times 10\ (4.11 - 1)\ 3 \div (\tfrac{3}{4} \times 4.11 + 2) = 1378 \text{ ft-lbs.}$$

83. Pivot Friction. — Let W be the load, μ the coefficient of friction, dimensions as shown in Figs. 161 and 162.

(i) On a flat pivot (Fig. 161) the average pressure per unit area of contact is $W/\pi R^2$. On any element of area dA the normal pressure is $(W/\pi R^2)dA$

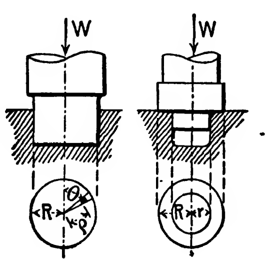

FIG. 161

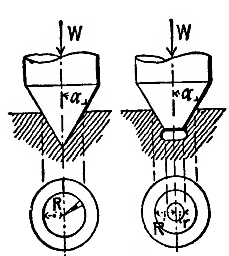

FIG. 162

(supposing that the total pressure is uniformly distributed), and the frictional resistance on the element is $\mu(W/\pi R^2)dA$. The moment of this resistance about the axis of the shaft is $\mu(W/\pi R^2)dA\rho$. We take $dA = \rho d\theta d\rho$; then the total resisting moment is

$$\int_0^{2\pi}\int_0^{r} \mu \frac{W}{\pi R^2} d\theta \rho^2 d\rho = \mu W \tfrac{2}{3} R. \quad \dots\dots \quad (1)$$

Thus the actual resistance may be regarded as a single force μW with an arm $\frac{2}{3}R$.

(2) For a collar bearing pivot (Fig. 161) the foregoing analysis holds but the ρ limits of integration would be r and R. The final expression for frictional resisting moment is

$$\mu W \frac{2}{3} \frac{R^3 - r^3}{R^2 - r^2}. \qquad (2)$$

Hence one may regard the resistance as a single force μW with an arm

$$\tfrac{2}{3}(R^3 - r^3) \div (R^2 - r^2).$$

(3) On a conical pivot (Fig. 162), the total normal pressure, and hence the friction too, is increased by wedge action. Let p be the intensity of normal pressure at any point of the contact, regarded as constant. Then the normal pressure on an elementary area $dA = pdA$. Since there is no tendency to slip vertically the friction has no vertical component, the vertical component of the normal pressures on all the elementary areas equals W; that is,

$$\int p\,dA \cdot \sin\alpha = W = pA \sin\alpha, \text{ or } p = \frac{W}{A \sin\alpha}.$$

But $A \sin\alpha$ is the area of the horizontal projection of the actual surface of contact. Hence the intensity of the normal pressure is independent of α, the pivot angle; p is $W/\pi R^2$; the normal pressure on the elementary area dA is $(W/\pi R^2)\,dA$, and the frictional resistance is $\mu(W/\pi R^2)\,dA$. The moment of the resistance about the axis of the shaft is $\mu(W/\pi R^2)\,dA\rho$, and the entire resisting moment is the integral of this expression. For simplicity in integration, imagine dA to be of such shape that its horizontal projection equals $\rho d\theta d\rho$. Then $\sin\alpha dA = \rho d\theta d\rho$, and the resisting moment is

$$\int_0^{2\pi} \int_0^R \frac{\mu W\,d\theta \rho^2\,d\rho}{\pi R^2 \sin\alpha} = \frac{\mu W}{\sin\alpha} \frac{2}{3} R. \qquad (3)$$

Hence one may regard the resistance as a single force $\mu W/\sin\alpha$ with an arm $\frac{2}{3} R$.

(4) For a frustrated conical pivot (Fig. 162), the analysis under (3) holds except that the ρ limits of integration are r and R. The final expression for frictional resisting moment is

$$\frac{\mu W}{\sin\alpha} \frac{2}{3} \frac{R^3 - r^3}{R^2 - r^2}. \qquad (4)$$

Hence one may regard the resistance as a single force $\mu W/\sin\alpha$ with an arm $\frac{2}{3}(R^3 - r^3) \div (R^2 - r^2)$.

When the pivots are actually turning then the above expressions for frictional moments still hold if μ is understood to mean coefficient of kinetic friction (Art. 215).

84. Rolling Resistance. — When a wheel or roller is made to roll, it experiences more or less resistance from the roadway or track upon which

it rolls. Obviously the amount of this resistance depends in large part on the nature of the surfaces of contact and on the amount of the pressure between them. In the case of an inelastic roadway (A, Fig. 163) the wheel leaves a rut, and in the case of an elastic roadway no rut is left. In either case there is more or less slipping and hence friction between the parts of the wheel and roadway in contact.

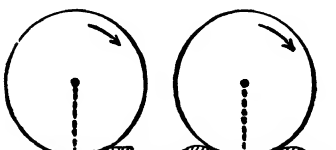

FIG. 163

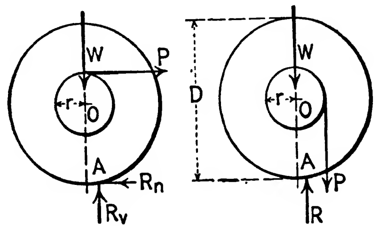

FIG. 164

Obviously the point of application of the resultant reaction of the roadway on the wheel is on the surface (or arc) of contact between them; and it would seem that this point is in front of the vertical diameter of the wheel, the roadway being supposed to be horizontal. The distance from this point to the diameter when the wheel is about to roll or is rolling at constant speed is called the *coefficient of rolling resistance.* We denote this coefficient by c and express numerical values of it in inches.

Let either part of Fig. 164 represent an isolated pair of railway car wheels, on rails, just about to roll by reason of a pull P applied by means of a cord midway between the rails wrapped about the axle (size much exaggerated). A represents the points of application of the reactions of both rails. Let R denote the total reaction on both wheels and W the entire weight; then the forces W, P and R may be regarded as coplanar. The moment of this system of forces about any point in the plane equals zero; and hence whether P is applied as shown in the figure or not, the

$$\text{“ driving moment ” (of } P \text{ about } A) = Wc.$$

If the coefficient c is independent of the way in which P is applied then the driving moment is likewise independent, and may appropriately be taken as a measure of the rolling resistance.

If the pull P is horizontal, then $P(\frac{1}{2}D + r) = Wc$, and

$$P = \frac{c}{\frac{1}{2}D + r}W,$$

which is also the value of the horizontal component R_h of the reaction of the roadway. This force R_h is sometimes called the rolling resistance but since it depends on r, the term thus applied seems inappropriate.

If the pull P is vertical then $P(r - c) = Wc$, or

$$P = \frac{c}{r - c}W$$

In this case there is no horizontal component of the reaction R; hence no rolling resistance in the sense last mentioned.

If the force P is horizontal and applied to the center of the wheel (Fig. 165) as in a horse-drawn vehicle,

$$R_h = P = \frac{c}{\frac{1}{2}D}W.$$

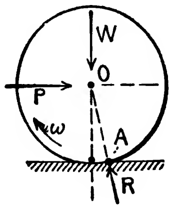

FIG. 165

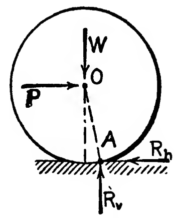

FIG. 166

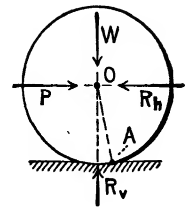

FIG. 167

Here W means weight of wheel plus load supported by that wheel; axle friction is neglected (see Art. 217). In Figs. 166 and 167 R is shown replaced at A and O respectively by its horizontal and vertical components. Either R_h may be regarded as the rolling resistance.

Rollers are generally used somewhat as shown in Fig. 168, so that there is rolling at top and bottom. Thus there is a pressure or reaction at top and bottom of each roller. If the weight of the roller is negligible then the reactions are equal and colinear. If the coefficients at top and bottom are equal, the reactions act through the center of the roller (Fig. 169) and

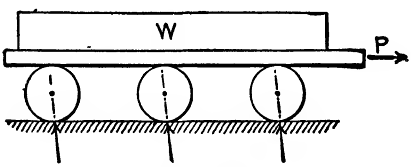

FIG. 168

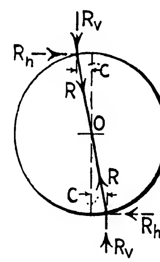

FIG. 169

$$R_h 2\sqrt{\tfrac{1}{4}D^2 - c^2} = R_v 2c,$$

or

$$R_h = \frac{c}{\sqrt{\frac{1}{4}D^2 - c^2}}R_v \approx \frac{c}{\frac{1}{2}D}R_v.$$

Let R', R'', etc. be the reactions of the roadway on the several rollers and assume the coefficient c the same at all places. Then

$$R_h' = \frac{c}{\frac{1}{2}D}R_v', \quad R_h'' = \frac{c}{\frac{1}{2}D}R_v'', \text{ etc.};$$

hence

$$(R_h' + R_h'' + \ldots) = \frac{c}{\frac{1}{2}D}(R_v' + R_v'' + \ldots);$$

or

$$P = \frac{c}{\frac{1}{2}D}W.$$

The laws of rolling resistance are not well known. Probably they are not simple. The coefficient certainly depends on the kind of wheel (or roller) and roadway; it may depend also on diameter of wheel (or roller) width of tire (or length of roller), and the load. In some investigations the coefficient did not vary much with diameter and load (for the ranges of diameter and load experimented with); in others the coefficient varied about as the square root of the diameter. Apparently no tests have shown much influence of load on the coefficient. Values of the coefficient, experimentally determined, are few in number and more or less uncertain. We give some below. (In traction analysis for highway and railway transport it is convenient to regard rolling and axle resistances — and even air resistances — as combined into a single equivalent resistance. Abundant information about such combined resistances is available, especially in recent publications[1] on highway and railway transport.)

Coefficient of Rolling Resistance c, in Inches

Lignum Vitae on oak	0.017 to 0.020
Elm on oak	0.03
Iron on iron	0.018 to 0.020

The first values were obtained with rollers 2 to 6 inches in diameter and load from 220 to 2200 pounds; the second with rollers 6 to 13 inches in diameter; the third with railroad wheels 20 and 40 inches in diameter.

For some conditions the coefficient seems to vary as the square root of the radius of the roller, that is

$$c = \phi \sqrt{\tfrac{1}{2} D}$$

where ϕ is another coefficient. Prof. C. L. Crandall[2] has established the following values.

Rollers	*Cast Iron*	*Steel*	*Wrought Iron*
$\phi =$	0.0063	0.0073	0.0120

for cast iron plates or roll ways. For steel plates the values are about 13 per cent less and for wrought iron about 13 per cent greater. Roller plates used were $1\frac{1}{2}$ inches thick; rollers 1, 2, 3 and 4 inches in diameter, all $1\frac{1}{2}$ inches long except the first whose length was 1 inch. Plates and rollers were used as they came from the plane and lathe; were not polished or filed. Loads varied from 350 to 2500 pounds per linear inch in contact.

[1] For example, University of Illinois Engineering Experiment Station Bulletin No. 110 on "Passenger Train Resistance" and Iowa State College Engineering Experiment Station Bulletin No. 67 on "Tractive Resistance and Related Characteristics of Roadway Surfaces."

[2] *Trans. Am. Soc. C. E.*, Vol. 32, p. 99 (1894).

Figure 170 represents in principle a device used to determine the coefficient of rolling resistance. W = weight of roller, W_1 and W_2 = weights of suspended bodies as shown. By choosing W_1 and W_2 any total load $(W + W_1 + W_2)$ can be put on the roller. By adjusting the difference between W_1 and W_2 the roller can be made to roll quite uniformly. When the roller is at rest or rolling at constant speed, the forces acting on it are in equilibrium (so that R is vertical as shown) and the moment of the entire system about any point equals zero. Thus for the point where R intersects the rim of the wheel

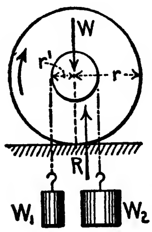

FIG. 170

$$Wc + W_1(r' + c) - W_2(r' - c) = 0, \quad \text{or}$$
$$c = \frac{W_2 - W_1}{W + W_1 + W_2} r',$$

from which c can be computed for any experimental values of W_1 and W_2.

Figure 171 represents in principle the device used by Crandall. There were two rollers under load — a third one to preserve stability only — and three plates as shown. The lower plate was supported on the weighing table of a testing machine; a large load was applied on the upper plate; and then the middle plate was subjected to a force P sufficient to start the plate. Thus the middle plate was subjected to the reactions of the two main rollers, inclined as shown. Let R = these reactions (nearly equal), θ = their inclination to the vertical, W = load, and D = diameter of rollers.

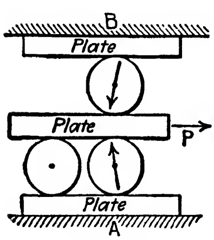

FIG. 171

Then evidently

$$2\,R \sin\theta = P, \quad \text{and} \quad R\cos\theta = W.$$

Hence $$P = 2\,W \tan\theta \approx 2\,W \frac{c}{\frac{1}{2}D}, \quad \text{or} \quad c = \frac{1}{4}\frac{P}{W}D,$$

from which c can be computed for any experimental values of P and W.

CHAPTER VI

CENTER OF GRAVITY; CENTROIDS

§ 1. Center of Gravity of a Body

85. Definition. — In Art. 6 it was stated that the weight of a body could, for certain purposes, be regarded as a concentrated force acting at a particular point called the *center of gravity* of the body. We now show why the weight can be so regarded, and subsequently, how the position of the center of gravity of a given body may be determined.

It is shown in Art. 30 that the resultant of two parallel forces F_1 and F_2 acting at two points A and B of any body cuts the line AB in a point P so that $AP/PB = F_2/F_1$. This proportion fixes the position of P, and since the proportion is independent of the angle between AB and the forces, P is also so independent. Therefore if AB (Fig. 172) were a rod and W_1 and W_2 the weights of two bodies suspended from A and B, then the resultant R of W_1 and W_2 would always pass through the same point even if the tilt of the rod were changed slowly so as to leave the suspending strings parallel. Furthermore, if three parallel forces be applied at definite points A, B and C of a body (Fig. 172), and if R denotes the resultant of W_1 and W_2 as before and R' the resultant of R and W_3 (and so also the resultant of W_1, W_2 and W_3), then $CP'/PP' = R/W_3$. This proportion fixes P' (in CP), and it is independent of the angle between the forces and the plane of ABC. Therefore if AB and CP be two rods rigidly fastened at P, and W_1, W_2 and W_3 the weights of bodies suspended from A, B and C, then the resultant of the three forces would always pass through P' if the rods were slowly turned about leaving the strings parallel. And so if any number of parallel forces have definite points of application on a rigid body, the resultant of the forces always passes through some one definite point of the body, or of its extension, when the body is turned about so as not to disturb the parallelism of the forces. This unique point is called the *center* or *centroid* of the parallel forces.

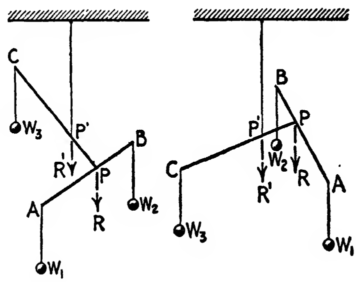

Fig. 172

The forces of gravity on all the constituent particles of a body constitute a parallel force system having definite points of application; therefore all

those forces have a centroid. That is, the resultant of the forces of gravity on all the particles of a body (its weight) always passes through some one definite point of the body, or of its extension, no matter how the body is turned about. It is this point that we call the center of gravity of the body.[1]

86. Center of Gravity Determined by Balancing. — Any single force that supports a body is in equilibrium with the weight of the body, hence must be colinear therewith, and so passes through the center of gravity of the body. This suggests the following way in which the position of the center of gravity of a given body may be determined by experiment: Determine the lines of action of two or more forces that, singly, will support the body; each of these lines of action must pass through the center of gravity, therefore the center of gravity is at their intersection.

For example, suppose it is desired to determine the position of the center of gravity of a thin plate of irregular form. The plate is suspended by a cord attached to it at any point A (Fig. 173); the axis of the string is the line of action of the supporting force, and this line is marked on the plate as indicated by AB. The plate is then suspended by a cord attached at any other point C and the line of action of the supporting force CD marked. The center of gravity of the plate is at O, the point of intersection of AB and CD.

Again, if a body is balanced on a knife-edge, the center of gravity is known to be in the vertical plane containing this knife-edge. If in some way the position of this plane in the body can be marked, one may, by successively balancing the body in different positions, determine three such planes, and, by their intersection, locate the center of gravity.

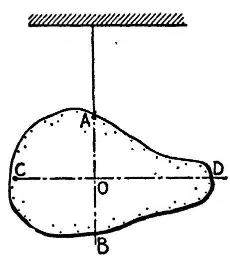

FIG. 173

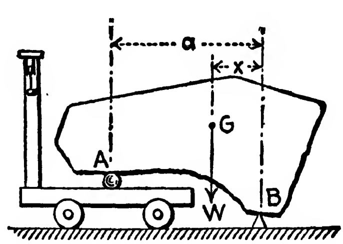

FIG. 174

87. Center of Gravity Determined by Weighing. — The weight W of the body is determined, and then it is supported on a knife-edge B (Fig. 174) and on a point support which rests upon a platform scale; the reaction W' of the point support is weighed, and the horizontal distance a of the point from the knife-edge is measured: Then the horizontal distance from the center of gravity to the knife-edge is $W'a/W$. In this manner the

[1] We call this point also the *center of mass*, or *mass center*, of the body.

horizontal distances of the center of gravity from several knife-edge supports can be got and the center of gravity located.

The distance of the center of gravity of a body from the plane through three points of the body can be determined if the body can be supported at the points and if certain weighings can be performed as described. Let A, B and C (behind B and not shown) be three such points of the body (Fig. 175); a = distance of A from the line joining B and C; W = weight of the body; W' = weight recorded by the scale when A, B and C are at the same level as shown in Fig. 175, and W'' = weight recorded

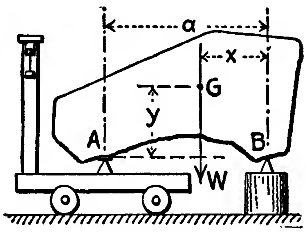

FIG. 175

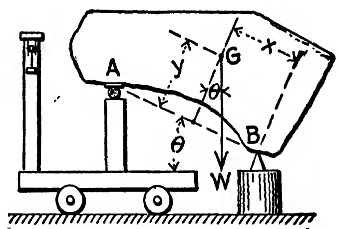

FIG. 176

by the scale when A is higher than B and C by any amount h (Fig. 176). Then the distance y of the center of gravity from the plane ABC is given by

$$y = \frac{\sqrt{a^2 - h^2}}{h} \frac{W' - W''}{W} a.$$

Proof: From the first position it is plain that $W'a = Wx$; from the second it follows that $W''a \cos \theta = W(x \cos \theta - y \sin \theta)$. Solving these simultaneously we get $y = (W' - W'') (a \cot \theta)/W$; but $\cot \theta = \sqrt{a^2 - h^2} \div h$, hence, etc.

88. Center of Gravity Determined by Inspection. — The notion of *balance*, brought out by the discussion of the preceding articles, suggests that for certain forms of homogeneous bodies the position of the center of gravity can be wholly or partially determined by inspection. We perceive instinctively that any homogeneous *symmetrical* solid will balance, as it were, on a plane of symmetry, and that therefore any plane or axis of symmetry must contain the center of gravity. Thus the center of gravity of a right parallelopiped, right circular cylinder, or sphere is seen to be at the geometrical center; the center of gravity of a right circular cone is seen to be on the axis and nearer the base than the apex, etc. This fact or condition of symmetrical balance is really the basis of most methods of determining the position of the center of gravity.

89. Composite Body; Determination of Center of Gravity. — By composite bodies we mean such as one naturally regards as consisting of parts. Thus most fly wheels are composite since they consist of a rim, a hub, and several spokes. If the parts of such a body are so simple that their weights

and centers of gravity are easily determinable, then the center of gravity of the entire body can be determined readily by means of the principle of moments as here explained.

Let A, B, C, etc. (Fig. 177), be the centers of gravity of certain parts of a body (not shown); W_1, W_2, W_3, etc., the weights of those parts; x_1, y_1, z_1 the coördinates of A; x_2, y_2, z_2, the coördinates of B, etc. Also let W denote the weight of the whole body, Q its center of gravity, and $\bar{x}$, $\bar{y}$, $\bar{z}$, the coördinates of Q. Since W is the resultant of W_1, W_2, W_3, etc., the moment of W about any line equals the algebraic sum of the moments of W_1, W_2, W_3, etc., about the same line (Art. 20). Thus, taking moments about the y-axis gives

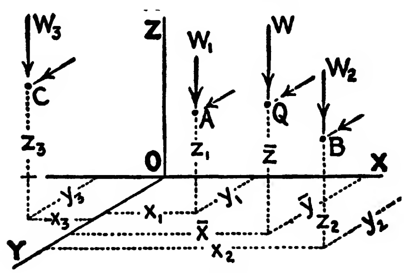

FIG. 177

$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 + \ldots,$$

from which equation $\bar{x}$ can be determined. Similarly, taking moments about the x-axis gives $\bar{y}$. To get $\bar{z}$, imagine the body turned until the y-axis is vertical, — the coördinate axes are assumed fixed to the body, — and then take moments about the x-axis; or, what comes to the same thing, imagine the forces of gravity (W_1, W_2, W_3, etc.) all turned about their respective points of application until they become parallel to the y-axis, and then take moments with respect to the x-axis.

The composite body may have a hole or notch on some other such void. One may regard such a body to have been without voids originally and later transformed into its actual form. For a body so regarded, the principle of moments may be stated as follows: About any line or axis the moment of the weight of the actual body is equal to the moment of the original body *minus* the moments of the weights of the material removed to make the voids.

EXAMPLE 1. A slender uniform wire 43 in. long is bent into the form represented by the heavy line in Fig. 178. It is required to determine the position of the center of gravity of this bent wire.

Solution: The weight of the wire per inch is taken as w, and x-, y- and z-axes are chosen as indicated. The moments of the straight portions of the wire with respect to the yz, xz and xy planes are computed, the computations being tabulated as in the schedule below. The weights of the several straight portions of the wire are listed under W; the coördinates of their respective centers of gravity under x, y and z; and their moments with respect to the yz, xz and xy planes under Wx, Wy and Wz respectively. The weight of the wire is 43 w; the moments of the whole wire about the yz, xz and xy planes are respectively 177.5 w, 148 w and 192 w. The coördinates of the center of gravity of the whole wire are therefore

$$\bar{x} = 177.5\,w \div 43\,w = 4.13 \text{ in.},$$
$$\bar{y} = 148\,w \div 43\,w = 3.44 \text{ in.},$$
$$\bar{z} = 192\,w \div 43\,w = 4.47 \text{ in.}$$

W	x	y	z	Wx	Wy	Wz
5 w	−2.5	0	8	−12.5 w	0 w	40 w
6	0.0	3	8	00.0	18	48
8	0.0	6	4	00.0	48	32
10	5.0	6	0	50.0	60	00
10	10.0	3	4	100.0	30	40
4	10.0	−2	8	40.0	−8	32
43 w				177.5 w	148 w	192 w

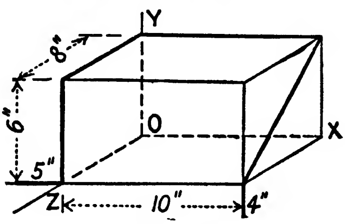

FIG. 178

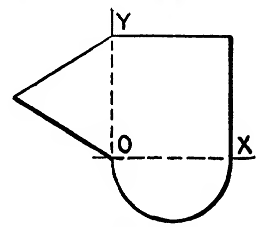

FIG. 179

EXAMPLE 2. Figure 179 represents a flat sheet of tin consisting of three parts, namely, a square, a semicircle, and an equilateral triangle. The square is 6 in. on a side. It is required to determine the position of the center of gravity of the sheet.

Solution: The weight of the tin is taken as w per sq. in. and x-, y- and z-axes are chosen as indicated (the z-axis is perpendicular to the plane of the paper). The moments of the several parts with respect to the yz and xz planes are computed, the computations being tabulated as in the schedule below.

For the square the center of gravity is obviously at the intersection of the diagonals. For a semicircle it is stated in Art. 94 that the center of gravity is $4\,r/3\,\pi$ distant from the center of the circle, where r is the radius. For a triangle it is stated in Art. 94 that the center of gravity is $1/3\,a$ distant from any base, where a is the altitude measured from that base. Hence the coördinates of the centers of gravity of the parts are as given under x and y. The moments of the parts with respect to the yz and xz planes are given under Wx and Wy respectively. The weight of the sheet is 65.7 w and the

Part	W	x	y	Wx	Wy
Square...........	36 w	3.0	3.0	108 w	108 w
Semicircle........	14.1 w	3.0	−1.272	42.3 w	− 17.94 w
Triangle..........	15.6 w	−1.734	3.0	−27.06 w	46.8 w
	65.7 w			123.24 w	136.86 w

moments of the sheet with respect to the yz and xz planes are respectively +123.24 w and +136.86 w. The coördinates of the center of gravity of the sheet are therefore

$$\bar{x} = 123.24\,w \div 65.7\,w = 1.88 \text{ in.},$$
$$\bar{y} = 136.86\,w \div 65.7\,w = 2.08 \text{ in.}$$

EXAMPLE 3. Figure 180 represents, in section, a cylinder of cast iron with a conical recess in one end and a cylindrical hole in the other. It is required to determine the position of the center of gravity.

Solution: Obviously the center of gravity of the cylinder is in the plane of the section shown; therefore the x and y coördinates only need be computed. Axes are chosen as indicated and the computations tabulated as in the preceding examples.

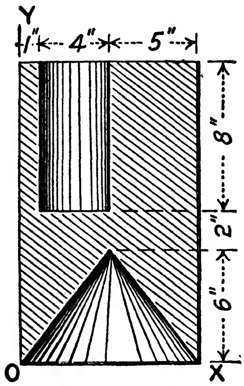

FIG. 180

The weights scheduled under W are the weight of the complete cylinder, the cone, and the small cylinder, all as of cast iron (specific weight 450 lbs. per cu. ft.). The weight of the cone and the weight of the small cylinder are considered negative, because these are parts removed.

The coördinates of the centers of gravity of the large and small cylinders are readily determined by inspection. For a cone it is shown in Art. 93 that the center of gravity is $1/4\ a$ distant from the base, where a is the altitude. Hence the coördinates of the centers of gravity of the parts are as given under x and y.

The moments of the parts are given in the last two columns.

Part	W	x	y	Wx	Wy
Solid cylinder...........	327.5	5	8	1637.5	2620.0
Cone....................	−41.0	5	1.5	−205.0	−61.5
Hole....................	−26.2	3	12	−78.6	−314.4
Cylinder as given.......	260.3			1353.9	2244.1

The weight of the cylinder as given is 260.3 lbs.; its moment with respect to the yz plane is 1353.9 in-lbs.; its moment with respect to the xz plane is 2244.1 in-lbs. The coördinates of its center of gravity are therefore

$$\bar{x} = 1353.9 \div 260.3 = 5.2 \text{ in.},$$
$$\bar{y} = 2244.1 \div 260.3 = 8.6 \text{ in.}$$

§ 2. Centroids of Lines, Surfaces and Solids

90. Definitions and Formulas. — Lines, surfaces, and (geometric) solids have no weight, and therefore they have no center of gravity in the strict sense of the term as defined in the preceding article. However, it is quite customary to speak of the center of gravity of these geometric conceptions, meaning by the term, the center of gravity of the line, surface or volume *materialized*, that is, conceived as a homogeneous slender wire, thin plate, or body, respectively. The center of gravity of a line, surface or solid is sometimes spoken of as the center of gravity of the length (of the line), area (of the surface), and volume (of the solid). The term *centroid* has been proposed as a substitute for center of gravity when applied to lines, surfaces and solids as being more appropriate; the new term is given preference in this book.

If a given line, surface, or solid is imagined as materialized, then one can apply the principle of moments to it as in Art. 89. Thus, if W = the weight of the whole materialized line, surface or solid, W_1, W_2, W_3, etc. =

the weights of all the parts into which we imagine it divided, $\bar{x}$ = the coördinate of the center of gravity of the whole with reference to some convenient reference plane, and x_1, x_2, x_3, etc. = the coördinates of the centers of gravity of all the parts, respectively, then

$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 + \cdots .$$

But the weights W, W_1, W_2, W_3, etc., are proportional to the respective lengths (L, L_1, L_2, L_3, etc.) or areas (A, A_1, A_2, A_3, etc.) or volumes (V, V_1, V_2, V_3, etc.), as the case may be; and therefore it follows from the preceding equations that

$$\begin{aligned} &\text{for lines,} & L\bar{x} &= L_1x_1 + L_2x_2 + L_3x_3 + \cdots , \\ &\text{for surfaces,} & A\bar{x} &= A_1x_1 + A_2x_2 + A_3x_3 + \cdots , \\ &\text{and for solids,} & V\bar{x} &= V_1x_1 + V_2x_2 + V_3x_3 + \cdots . \end{aligned}$$

Similar equations hold for y and z coördinates.

91. Principle of Moments for Lines, Surfaces and Solids. — The foregoing three equations can be rendered conveniently in a single statement or proposition by means of a new term which we now define. The *moment of a line, surface or solid with respect to a plane* is the product of the length of the line, area of the surface, or volume of the solid and the coördinate of the centroid of the line, surface or solid with respect to that plane. Thus each of the products in the above equations is a moment with respect to the yz plane; Lx is the moment of the line L, Ax is the moment of the surface A, and Vx is the moment of the solid V. The proposition or principle of moments, then, is this: The moment of a line, surface or solid with respect to any plane equals the algebraic sum of the moments of the parts of that line, surface or solid into which we imagine the whole divided, with respect to that same plane.[1] *Obviously the moment of a line, surface or solid with respect to any plane containing the centroid thereof is zero.*

If any given line or surface lies wholly in a plane, say the xy coördinate plane, then we call the products Lx and Ax respectively the moments of L and A with respect to the y coördinate axis. Similarly Ly and Ay are the moments of L and A with respect to the x-axis. Hence for a plane, line, or surface we may state this proposition: With respect to any coördinate axis in the plane the moment of the line or surface equals the sum of the moments of parts of that line or surface into which we imagine the whole divided. Obviously the moment of a line or surface is zero with respect to any axis containing the centroid thereof.

92. Composite Lines, Surfaces and Solids; Determination of Centroid. — By composite lines, surfaces or solids we mean those which one

[1] Of course these moments have nothing to do with turning effects like the moment of a force with respect to a line or a point. To distinguish these moments, the former are sometimes called *statical moments*, not very appropriately, however.

naturally regards as consisting of parts (see Figs. 181, 182 and 183). If the composite figure consists of *simple* parts whose lengths, areas or volumes (as the case may be) and centroids are known, then the centroid of the composite figure can be determined easily by means of the principle of moments as illustrated in examples 1, 2 and 3 below:

If the composite figure has void parts, then in applying the principle of moments, one *subtracts* the moments of the voids (see Ex. 4 below).

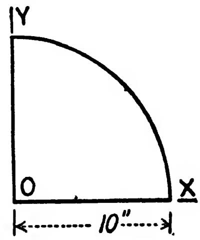

FIG. 181

EXAMPLE 1. It is required to locate the centroid of the line represented (heavily) in Fig. 181, the curved portion of which is a circular arc; given that each coördinate of the centroid of the arc is 6.366 in. (Art. 93).

Solution: Obviously the x and y coördinates of the line are equal, therefore moments are taken about one line only, the x-axis being chosen. The computations are tabulated in the schedule below, the length of the parts being given in the second column, the y coördinates of their centroids in the third, and the moments about the x-axis in the last.

Part	L	y	Ly
Vertical..........	10	5	50
Horizontal........	10	0	0
Arc...............	15.7	6.366	99.95
Whole	35.7		149.95

The y coördinate of the centroid of the whole line is

$$\bar{y} = 149.95 \div 35.7 = 4.20 \text{ in., and also}$$
$$\bar{x} = 4.20 \text{ in.}$$

EXAMPLE 2. It is required to locate the centroid of the shaded area in Fig. 182, which represents the cross section of a "channel" (a form of steel beam much used in onstruction).

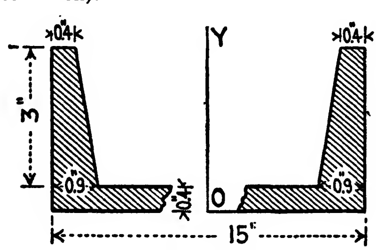

FIG. 182

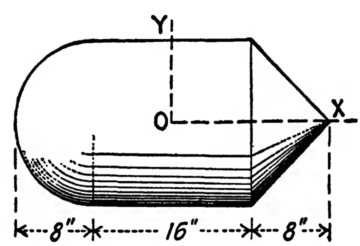

FIG. 183

Solution: The area is considered as divided into a rectangle (0.40 in. $\times$ 15 in.) and two trapezoids. The centroid of the rectangle is evident upon inspection· the distance of the centroid of either trapezoid from its longer base is given by

$$3\,(0.90 + 0.80) \div 3\,(0.90 + 0.40) = 1.31 \text{ in. (Art. 94).}$$

Since the centroid of the area is obviously on the vertical axis of symmetry (the y-axis) the y coördinate only of the centroid need be computed, and so moments are taken about

the x-axis only (base of figure). The computations are tabulated in the schedule below, the areas of the parts being given in the second column, the y-coördinates of their centroids in the third, and the moments in the last.

Part	A	y	Ay
Rectangle........	6.0	0.20	1.20
Two trapezoids...	3.9	1.71	6.67
Whole...........	9.9		7.87

The y coördinate of the centroid of the whole area is

$$\bar{y} = 7.87 \div 9.9 = 0.79 \text{ in.}$$

EXAMPLE 3. It is required to locate the centroid of a solid consisting of a cone, a cylinder and hemisphere as represented in Fig. 183, given that the centroid of the cone is 2 in. from its base, and that of the hemisphere is 3 in. from its base (Art. 94).

Solution: Obviously the centroid is on the longitudinal axis of the figure; therefore the x coördinate only need be computed. Moments are taken with respect to the yz plane, the computations being tabulated in the schedule below. The volumes of the parts are given in the second column, the x coördinates of their centroids in the third, and the moments in the last.

Part	V	x	M
Cone............	536.2	10	5,362
Cylinder.........	3217.0	0	0
Hemisphere.......	1072.3	−11	−11,795
	4825.5		−6,433

The x coördinate of the centroid of the whole solid is

$$\bar{x} = -6433 \div 4825.5 = -1.33 \text{ in.}$$

(The negative sign indicates that the centroid is to the left of 0.)

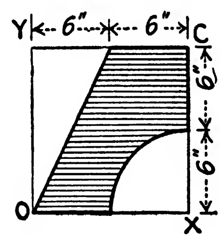

FIG. 184

EXAMPLE 4. It is required to locate the centroid of the shaded area in Fig. 184, the part of the square remaining after the triangle and the quadrant have been taken away; given that the centroid of the triangle is 2 in. from OY and 4 in. from YC, and that the centroid of the quadrant is 2.54 in. from OX and CX (Art. 94).

Solution: Moments are taken about the x- and y-axes. The areas, centroidal coördinates, and moments are given in the schedule below; the areas and moments of the parts taken away are subtracted from the area and moments of the square.

Part	A	x	y	Ax	Ay
Square................	144	6	6	864	864
Triangle...............	−36	2	8	−72	−288
Quadrant..............	−28.27	9.44	2.54	−266.9	−71.8
Shaded area...........	79.73			525.1	504.2

The x coördinate of the centroid of the shaded area is

$$\bar{x} = 525.1 \div 79.73 = 6.59 \text{ in.}$$

and the y coördinate is

$$\bar{y} = 504.2 \div 79.73 = 6.32 \text{ in.}$$

93. Centroids Determined by Integration. — If it is desired to locate the centroid of a line, a surface, or a solid which cannot be divided into a finite number of simple parts whose lengths, areas, or volumes and centroids are known, and if the line, surface, or solid is "mathematically regular," then one can imagine the line, surface, or solid divided into an infinitely great number of parts, and apply the principle of moments. To find the sum of the moments of all these elementary parts involves an integration. Thus, let L = the length of a line, $\bar{x}$ = the x coördinate of its centroid, dL_1, dL_2, dL_3, etc. = the lengths of elementary portions of the line, and x_1, x_2, x_3, etc. = the x coördinates of the centroids of those portions respectively,

then $$L\bar{x} = dL_1 \cdot x_1 + dL_2 \cdot x_2 + dL_3 \cdot x_3 + \cdots = \int dL \cdot x,$$

in which dL stands for any of the elementary lengths and x for the x coördinate of the centroid of that dL. Similarly, for areas and volumes; and thus we have these formulas:

$$(1)\ L\bar{x} = \int dL \cdot x; \quad (2)\ A\bar{x} = \int dA \cdot x; \quad (3)\ V\bar{x} = \int dV \cdot x,$$

and corresponding ones for $\bar{y}$ and $\bar{z}$ (the y and z coördinates of the centroid).

These formulas can be used to determine $\bar{x}$, $\bar{y}$, and $\bar{z}$ if the form of the line, surface, or solid is such that the integrations can be performed. In any particular case, limits of integration must be assigned so that all elementary portions are included in the integration (summation).[1] Six examples illustrating their use follow:

EXAMPLE 1. It is required to locate the centroid of a circular arc of radius = r and central angle = 2α (Fig. 185).

Solution: The radius which bisects the central angle is a line of symmetry, therefore the centroid is on that line; if that line is taken as x-axis, then $\bar{y} = 0$. The length of the arc = $2r\alpha$ (α expressed in radians), $dL = rd\phi$, and $x = r\cos\phi$; therefore formula (1) becomes

$$2r\alpha\bar{x} = \int_{-\alpha}^{+\alpha} r\,d\phi \cdot r\cos\phi = r^2 \int_{-\alpha}^{+\alpha} \cos\phi\, d\phi = 2r^2 \sin\alpha; \text{ or } \bar{x} = (r\sin\alpha) \div \alpha.$$

[1] *The centroid is a mean point.* The ordinate from any plane to the centroid of a line, surface, or solid equals the mean of the ordinates of all the equal elementary portions of the line, surface, or solid, it being understood that the mean takes into account signs of the ordinates. For, let x_1, x_2, x_3, etc., be the ordinates of the elementary portions and n the number of them (infinite); then the mean ordinate is $(x_1 + x_2 + x_3 + \cdots) \div n$; also, let Q = the length, area, or volume of the line, surface, or solid, and dQ = the length, area, or volume of the equal elementary portions; then the mean ordinate equals

$$\frac{(x_1 + x_2 + \cdots)dQ}{ndQ} = \frac{\int dQ \cdot x}{Q} = \bar{x}.$$

The preceding problem will now be solved without using polar coördinates. Since $x^2 + y^2 = r^2$, $x\,dx + y\,dy = 0$, or $dy = -(x\,dx)/y$. Hence

$$dL = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + x^2/y^2} = dxr/y = dxr/\sqrt{r^2 - x^2}.$$

and
$$2\,r\alpha\bar{x} = \int x dL = 2\,r\int_{r\cos\alpha}^{r} \frac{x\,dx}{\sqrt{r^2 - x^2}} = 2\,r^2\sin\alpha;\text{ etc.}$$

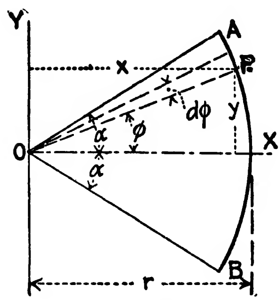

FIG. 185

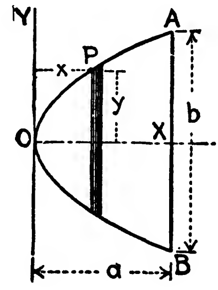

FIG. 186

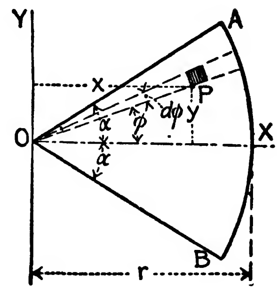

FIG. 187

EXAMPLE 2. It is required to locate the centroid of the parabolic segment $AOBA$ (Fig. 186); altitude $= a$ and base $= b$.

Solution: Evidently the axis of the parabola is a line of symmetry, and therefore it contains the centroid. If that line be taken as the x-axis, then $\bar{y} = 0$. Let x and y be the coördinates of any point P on the parabola; then the area of the elementary portion shaded is $2\,y\,dx$. Since the area of the segment is $\frac{2}{3}\,ab$, and the equation of the parabola is $4\,ay^2 = b^2x$, formula (2) becomes

$$\frac{2}{3}\,ab\bar{x} = \int_0^a 2\,y\,dx \cdot x = \frac{b}{\sqrt{a}}\int_0^a x^{\frac{3}{2}}\,dx = \frac{2}{5}\,ba^2;$$

and
$$\bar{x} = \frac{2}{5}\,ba^2 \div \frac{2}{3}\,ab = \frac{3}{5}\,a.$$

EXAMPLE 3. It is required to locate the centroid of the circular sector (Fig. 187); radius $= r$ and central angle $= 2\,\alpha$.

Solution: The radius which bisects the central angle is evidently a line of symmetry, and so the centroid is on that line. If that line is taken as x-axis, then $\bar{y} = 0$. The area of the sector equals $r^2\alpha$, α expressed in radians; $dA = \rho\,d\phi \cdot d\rho$, where $\rho = OP$ and P is any point in the sector. Therefore formula (2) becomes

$$r^2\alpha\bar{x} = \int_0^r\int_{-\alpha}^{+\alpha} \rho\,d\phi\,d\rho \cdot x = \int_0^r\int_{-\alpha}^{+\alpha} \rho\,d\phi\,d\rho \cdot \rho\cos\phi;$$

and
$$\bar{x} = \left(\frac{2}{3}\,r^3\sin\alpha\right) \div \left(r^2\alpha\right) = \frac{2\,r\sin\alpha}{3\,\alpha}.$$

EXAMPLE 4. It is required to locate the centroid of a conical or pyramidal solid; altitude $= a$ (Fig. 188).

Solution: The origin of coördinates is taken at the apex, and the x-axis perpendicular to the base; $OMNO$ represents the projection of the cone or pyramid on the XY plane. The solid is imagined divided into plates or laminas parallel to the base; if the area of the base is called A, say, then the area of the lamina represented is Ax^2/a^2, and the volume of the lamina is $dx \cdot Ax^2/a^2$. And since the volume of the solid is $\frac{1}{3}\,Aa$, formula (3) becomes

$$\frac{1}{3}\,Aa\bar{x} = \int_0^a (dx \cdot Ax^2/a^2)\,x = A/a^2\int_0^a x^3dx = \frac{Aa^2}{4};$$

hence,

$$\bar{x} = \frac{3}{4} a,$$

that is, the perpendicular distance from the centroid to the base equals one-fourth the altitude. Evidently, the centroid of every lamina lies on the line joining the apex and the centroid of the base; therefore the centroid of all the laminas (that is, of the solid) lies on that line.

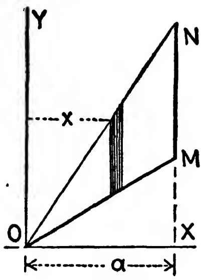

FIG. 188

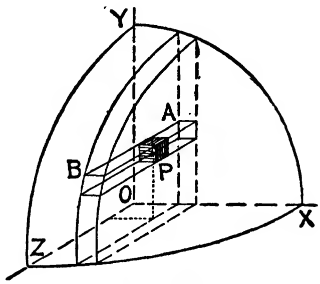

FIG. 189

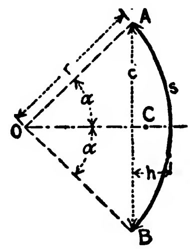

FIG. 190

EXAMPLE 5. It is required to locate the centroid of the octant of a sphere; radius $= r$ (Fig. 189).

Solution: Obviously $\bar{x} = \bar{y} = \bar{z}$; $\bar{x}$ is given by

$$V\bar{x} = \int_0^r \int_0^{(r^2 - x^2)^{\frac{1}{2}}} \int_0^{(r^2 - x^2 - y^2)^{\frac{1}{2}}} (dx\, dy\, dz)x.$$

Evaluating the integral and substituting for V its value, $\frac{1}{6}\pi r^3$, it is found that

$$\bar{x} = \tfrac{3}{8} r.$$

94. Centroids of Some Lines, Surfaces and Solids. — *Circular Arc* (Fig. 190). — C is the centroid; its distance from the center is $(r \sin \alpha)/\alpha$, the divisor α to be expressed in radians (1 degree = 0.0175 radian); the distance is also rc/s, where s = arc. If the arc is flat then the distance of its centroid from the chord is nearly $\frac{2}{3} h$; the discrepancy is less than one-half per cent for arcs whose central angle 2α is less than 60 degrees.

When the arc is a semicircle, then the distance from the centroid to the center is $2r/\pi = 0.6366\, r$. When the arc is a quadrant, then the distance to the center is $2r\sqrt{2}/\pi = 0.9003\, r$, and the distance to the radii OA and OB is $2r/\pi = 0.6366\, r$.

Triangle. — The centroid is at the intersection of the medians; its perpendicular distance from any side equals one-third the altitude of the triangle measured from that side. If x_1, x_2, and x_3 are the coördinates of the vertexes with respect to any plane and $\bar{x}$ the coördinate of the centroid, then $\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$.

Trapezoid. — The centroid is on the median (line joining the middle points of the parallel sides) (Fig. 191),

$$l = \frac{(B - b)a}{6(B + b)}, \quad m = \frac{(2B + b)a}{3(B + b)}, \quad n = \frac{(B + 2b)a}{3(B + b)}.$$

Two geometrical constructions for locating position on the median follow: (1) Extend AE (Fig. 192) so that $BE = CD$, and in the opposite direction

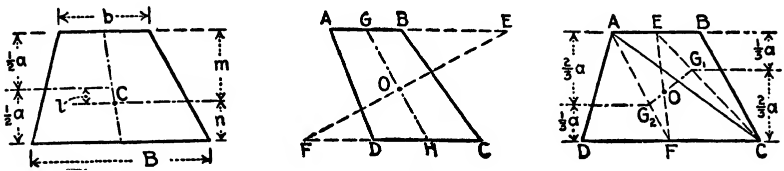

FIG. 191 FIG. 192 FIG. 193

extend CD so that $DF = AB$; the intersection of FE and the median GH is the centroid sought. (2) Divide the trapezoid (Fig. 193) into triangles by a diagonal as AC; find the centroids G_1 and G_2 of the triangles (construction indicated in the figure); the intersection G_1G_2 with the median EF is the centroid sought.

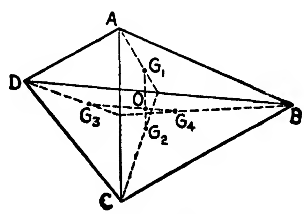

FIG. 194

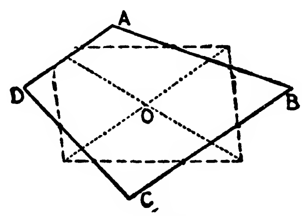

FIG. 195

Quadrilateral. — (1) Divide the quadrilateral into triangles by a diagonal DB (Fig. 194) and find their centroids G_1 and G_2; divide it into triangles by the other diagonal and find their centroids G_3 and G_4; the intersections of the lines G_1G_2 and G_3G_4 is the centroid sought. (2) Divide the sides into thirds (Fig. 195), and draw lines through the third points as shown; these lines form a parallelogram whose diagonals intersect at the centroid of the quadrilateral.

Sector of a Circle (Fig. 196). — C is the centroid; its distance from the center is $\frac{2}{3}(r \sin \alpha)/\alpha$, the divisor α to be expressed in radians (1 degree = 0.0175 radian); the distance also equals $\frac{2}{3} rc/s$, where s = arc.

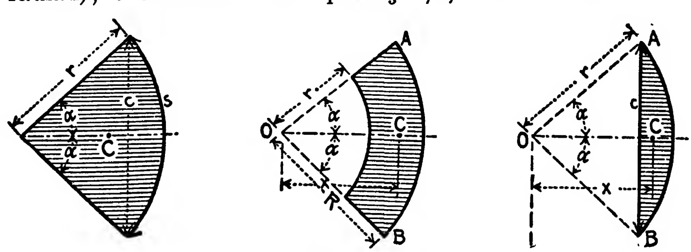

FIG. 196 FIG. 197 FIG. 198

When the sector is a quadrant, then the distance of the centroid from the center is $4\sqrt{2}\, r/3\pi = 0.6002\, r$; and the distance to the radii OA and OB

is $4\,r/3\,\pi = 0.4242\,r$. For a semicircle the distance from diameter to centroid is $4\,r/3\,\pi = 0.4242\,r$.

Sector of a Circular Ring (Fig. 197). — The distance from the centroid to the center is

$$\frac{2}{3}\,\frac{R^3 - r^3}{R^2 - r^2}\,\frac{\sin\alpha}{\alpha},$$

the divisor α to be expressed in radians (1 degree = 0.0175 radian).

Segment of a Circle (Fig. 198). — The distance from the centroid to the center is

$$\frac{c^3}{12\,A} = \frac{2\,r^3 \sin^3\alpha}{3\,A},$$

where A denotes the area of the segment. $A = \frac{1}{2}\,r^2\,(2\,\alpha - \sin 2\,\alpha)$, the first α to be expressed in radians (1 degree = 0.0175 radian).

The Area Shaded in Fig. 199, included between a quadrant and the

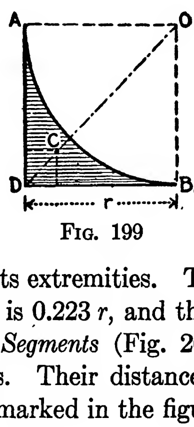

FIG. 199

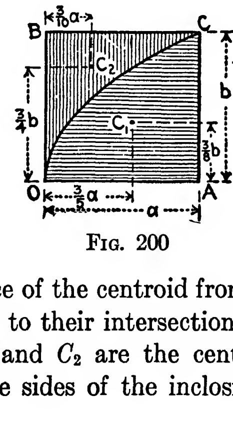

FIG. 200

tangents at its extremities. The distance of the centroid from the bounding tangents is $0.223\,r$, and the distance to their intersection is $0.315\,r$.

Parabolic Segments (Fig. 200). — C_1 and C_2 are the centroids of the shaded parts. Their distances from the sides of the inclosing rectangle ($a \times b$) are marked in the figure.

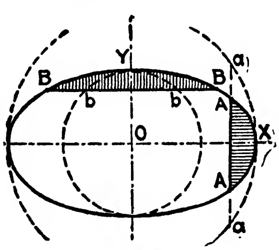

FIG. 201

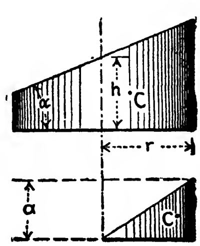

FIG. 202

Elliptic Segment (Fig. 201). — The centroid of the segment $XAAX$ coincides with the centroid of the segment $XaaX$ of the circumscribed circle; the centroid of the segment $YBBY$ coincides with the centroid of the $YbbY$ of the inscribed circle.

Right Circular Cylinder (Fig. 202). — C is the centroid; its distance from the axis of the cylinder is $\frac{1}{4}\,(r^2 \tan\alpha)/h$, and its distance from the base is

$\frac{1}{2}h + \frac{1}{8}(r^2 \tan^2 \alpha)/h$. When the oblique top cuts the base in a diameter of the base (lower part of Fig. 202), then the distance from the centroid to the axis is $\frac{3}{16}\pi r$, and to the base $\frac{3}{32}\pi a$.

Cone and Pyramid. — The centroid of the surface (not including base) is on a line joining the apex with the centroid of the perimeter of the base at a distance of two-thirds the length of that line from the apex. The centroid of the solid cone or pyramid is on the line joining the apex with centroid of the base at a distance of three-fourths the length of that line from the apex.

Frustum of a Circular Cone. — Let R = radius larger base, r = radius smaller, a = altitude. The distance of the centroid of the curved surface from larger base is $\frac{1}{3}a(R + 2r)/(R + r)$; from smaller base $\frac{1}{3}a(2R + r)/(R + r)$; from a plane midway between bases $\frac{1}{6}a(R - r)/(R + r)$. The distance from the centroid of the solid frustum to the larger base is

$$\frac{1}{4}a(R^2 + 2Rr + 3r^2)/(R^2 + Rr + r^2).$$

Frustum of a Pyramid. — If the frustum has regular bases, let R and r be the lengths of sides of the larger and smaller bases, and h the altitude; then the distance of the centroid of the surface (not including bases) from the larger base is $\frac{1}{3}h(R + 2r)/(R + r)$. Whether the bases are regular or not, let A and a = the areas of the large and small bases and h the altitude; then the distance of the centroid of the solid from the larger base is

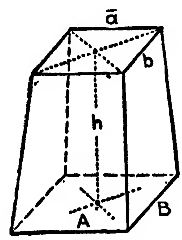

FIG. 203

$$\frac{1}{4}h(A + 2\sqrt{Aa} + 3a)/(A + \sqrt{Aa} + a).$$

Obelisk and Wedge (Fig. 203). — The distance from the centroid to the base AB is

$$\frac{1}{2}h(AB + Ab + aB + 3ab)/(2AB + Ab + aB + 2ab).$$

If $b = 0$ the solid is a wedge, and the distance from the centroid to the base is

$$\frac{1}{2}h(A + a)/(2A + a).$$

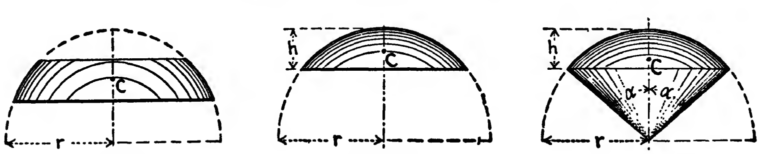

FIG. 204 FIG. 205 FIG. 206

Sphere. — The centroid of any zone (surface) of a sphere (Fig. 204) is midway between the bases. The distance of the centroid of a segment (solid) (Fig. 205) from the base is $\frac{1}{4}h(4r - h)/(3r - h)$; when $h = r$ (hemisphere) then the distance is $\frac{3}{8}r$. The distance of the centroid

of a sector (solid) (Fig. 206) from the center of the sphere is $\frac{3}{8}(1 + \cos \alpha)r = \frac{3}{8}(2r - h)$.

Ellipsoid. — Let the three axes be taken as x, y, and z coördinate axes, and let a, b, and c denote the semi-lengths of the corresponding axes of the ellipsoid; the centroid of one octant of the solid is given by $\bar{x} = \frac{3}{8}a$, $\bar{y} = \frac{3}{8}b$, and $\bar{z} = \frac{3}{8}c$.

Paraboloid of Revolution, formed by revolving a parabola about its axis. Let h = height of the paraboloid, the distance from its apex to the base, then the distance from the centroid of the solid to the base is $\frac{1}{3}h$.

CHAPTER VII

SUSPENDED CABLES (WIRE, CHAIN, ETC.)

95. Parabolic Cable; Symmetrical Case. — When a cable is suspended from two points and it sustains loads uniformly spaced along the horizontal and spaced so closely that the loading is practically continuous, then the curve assumed by the cable is a parabolic arc as will now be shown. The symmetrical case (points of suspension at same level) will be considered first. Let AOB (Fig. 207) be the cable suspended from A and B, w = load per unit (horizontal) length, a = span AB, f = sag, H = tension in cable at lowest point, and T = tension at any other point Q (coördinates x and y). The forces acting on the portion OQ are H, T, and the distributed load wx (Fig. 208); this load acts at mid-length of x. Since the forces are in equilibrium, their moment-sum equals zero for any origin of moments; hence moments about Q give $Hy = wx(x/2)$,

$$x^2 = \frac{2\,H}{w}y, \qquad \text{or} \qquad y = \frac{w}{2\,H}x^2. \quad \dots\dots \quad (1)$$

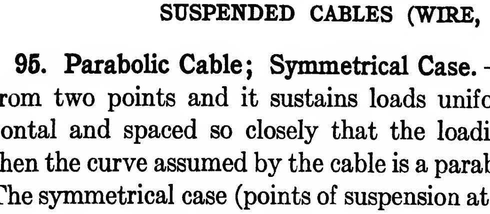

FIG. 207

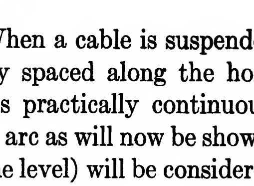

FIG. 208

This is the standard form of the equation of a parabola; the axis of the parabola coincides with the y-axis, and the vertex is at O. Substitution for x and y of their values for the point A ($x = a/2$, and $y = f$), gives $a^2/4 = (2\,H/w)f$, or $H = wa^2/8f$; hence equation (1) may be written

$$x^2 = \frac{a^2}{4f}y, \qquad \text{or} \qquad y = \frac{4f}{a^2}x^2. \quad \dots\dots\dots \quad (2)$$

A formula for the tension T at any point Q may be arrived at as follows: Let ϕ = slope of the curve at Q; then it is plain from Fig. 208 that $T \sin \phi = wx$, and $T \cos \phi = H$. Squaring and adding gives

$$T^2 = w^2x^2 + H^2 = w^2x^2 + w^2a^4/64f^2 \quad \dots\dots \quad (3)$$

At the points of suspension, $x = a/2$, and the value of T at that point is

$$H\left(1 + 16\frac{f^2}{a^2}\right)^{\frac{1}{2}} = \tfrac{1}{2}\,wa\left(1 + \frac{a^2}{16f^2}\right)^{\frac{1}{2}}. \quad \dots\dots\dots \quad (4)$$

The adjoining table gives values of T/wa for various values of f/a, the sag ratio (denoted by n in the table).

$n =$	1.0	0.5	0.25	0.125	0.1	0.05	0.01
$T/wa =$	0.515	0.559	0.707	1.118	1.346	2.550	12.81

The *length of cable* for any span a and sag f or sag ratio $n = f/a$. — Let l = length of cable AB and ds = length of an elementary portion; then, as in all plane curves, $ds^2 = dx^2 + dy^2 = [1 + (dy/dx)^2]\,dx^2$, or

$$ds = [1 + (dy/dx)^2]^{\frac{1}{2}}\,dx.$$

From the equation of the curve (2), $dy/dx = (8\,f/a^2)\,x = 8\,nx/a$; hence

$$ds = (1 + 64\,n^2x^2/a^2)^{\frac{1}{2}}\,dx.$$

Integrating between proper limits (0 and $\frac{1}{2}\,l$ for s, and 0 and $\frac{1}{2}\,a$ for x) and then doubling, gives

$$l = a\left(\tfrac{1}{2}\,(1 + 16\,n^2)^{\frac{1}{2}} + \frac{1}{8\,n}\log_e[4\,n + (1 + 16\,n^2)^{\frac{1}{2}}]\right) \quad \ldots \quad (5)$$

An approximate formula for l, much more convenient to use than the foregoing one, may be deduced as follows: Expanding the coefficient of dx above by the binomial theorem gives

$$ds = \left(1 + 32\,n^2\frac{x^2}{a^2} - 512\,n^4\frac{x^4}{a^4} + \cdots\right)dx;$$

and integrating between limits as before it is found that

$$l = a\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \cdots\right) \quad \ldots\ldots\ldots \quad (6)$$

The following table gives values of l/a by the exact and approximate formulas, equations (5) and (6) respectively, for several sag ratios $n = f/a$.

$n =$	1.0	0.5	0.25	0.125	0.1	0.05	0.01
l/a exact	2.3234	1.4789	1.1478	1.0402	1.0260	1.0066	1.0003
l/a approximate			1.1417	1.0401	1.0260	1.0066	1.0003

96. Parabolic Cable; Unsymmetrical Case. — By this is meant a cable suspended from two points not at the same level; see Fig. 209, where ACB represents a cable suspended from A and B. In this case, also, the cable hangs in the arc of a parabola as will be proved presently. Let a = horizontal distance between points of supports (as in Art. 95), b = vertical distance between the points, θ = angle which AB makes with the horizontal ($= \tan^{-1} b/a$), x and y = coördinates of any point Q of the cable as shown, T = tension at the highest point, V' and H' respectively = the two components of T along AY and AB. There are three forces acting on the

part AQ, — its load wx, the tension T, and the tension at Q. The moment-sum for these three forces for any origin equals zero; with Q as origin

$$-wx \cdot x/2 + V'x - H'(QP) = 0, \text{or } V'x - wx^2/2 = H'(y \cos \theta - x \sin \theta). \quad (1)$$

This is the equation of a parabola with the axis parallel to the y coördinate axis.

To express the equation of the curve in terms of the dimensions a, b, and the vertical sag f_1 under the middle point of the chord AB: — The forces acting on the entire cable consist of the load wa, the tension at A, and that at B. Their moment-sum with origin at B is

$$wa \cdot a/2 - V'a = 0;$$

hence $\quad V' = wa/2. \quad . \quad . \quad (2)$

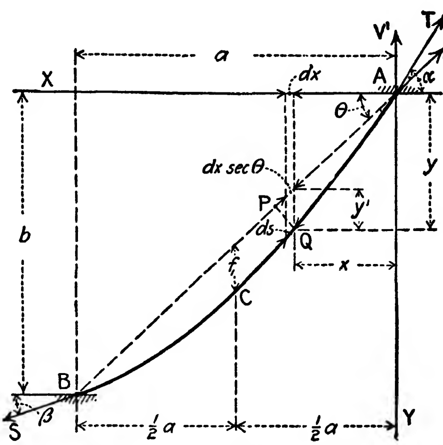

FIG. 209

The forces on the upper half AC consist of the load $wa/2$, the tension at A, and that at C. Their moment-sum with origin at C is

$$\frac{wa}{2}\frac{a}{4} - V'\frac{a}{2} + H'f_1 \cos \theta = 0; \quad \text{hence} \quad H' = \frac{wa^2}{8 f_1 \cos \theta}. \quad . \quad . \quad (3)$$

Substituting these values of V' and H' in (1) gives

$$\frac{4 f_1 x}{a^2}(a - x) = y - x \tan \theta, \quad \text{or} \quad y = (4 f_1 + b)\frac{x}{a} - 4 f_1 \frac{x^2}{a^2}. \quad . \quad (4)$$

The vertical distance of any point as Q below the chord AB is $y - x \tan \theta$; hence if we let y' denote that distance, the foregoing equation can be put into the more convenient form

$$y' = \frac{4 f_1 x}{a^2}(a - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of the slope at any point of the curve is (differentiating equation (4))

$$\frac{dy}{dx} = 4\frac{f_1}{a} + \frac{b}{a} - \frac{8 f_1}{a}\frac{x}{a}.$$

Let α and β = the slope angles at A and B respectively (where $x = 0$, and $x = a$); then

$$\tan \alpha = (b + 4 f_1)/a, \quad \text{and} \quad \tan \beta = (b - 4 f_1)/a \quad . \quad . \quad . \quad . \quad (6)$$

Let x_0 and y_0 = the coördinates of the vertex of the parabola (where $dy/dx = 0$); then

$$x_0 = a(b + 4f_1)/8f_1, \quad \text{and} \quad y_0 = (b + 4f_1)^2/16f_1. \quad . \quad . \quad . \quad (7)$$

Let H and V respectively = the horizontal and vertical components of T. Then (see Fig. 209)

$$H = H' \cos\theta = wa^2/8f_1, \text{ and}$$
$$V = V' + H' \sin\theta = \tfrac{1}{2} wa\,(1 + a\tan\theta/4f_1);$$

and since $T^2 = H^2 + V^2$, it is found that

$$T = \frac{wa^2}{8f_1}\left[1 + \left(\tan\theta + \frac{4f_1}{a}\right)^2\right]^{\frac{1}{2}} = \tfrac{1}{2} wa\left[\frac{1}{16n_1^2} + \frac{\sin\theta}{2n_1} + 1\right]^{\frac{1}{2}}, \quad . \quad (8)$$

where n_1 = sag ratio $f_1 \div AB = f_1 \div a\sec\theta$. The last expression shows that for given w, a, and n_1, the tension T increases as the angle θ is made larger; also that for given w, a, and θ, T increases as n_1 is made smaller.[1]

Length of the Parabolic Arc AB (Fig. 209). — Let a_1 = the length of the chord AB, n_1 = sag ratio $f_1 \div a_1$, and l_1 = length of the arc AB. Also let ds = length of an elementary portion of the arc; then

$$ds = [1 + (dy/dx)^2]^{\frac{1}{2}}\,dx.$$

From the equation of the curve (4), it follows that

$$\frac{dy}{dx} = \frac{4f_1}{a}\left(1 - \frac{2x}{a}\right) + \tan\theta = \left[4n_1\left(1 - \frac{2x}{a}\right) + \sin\theta\right]\sec\theta. \quad . \quad (9)$$

This last value of dy/dx substituted in the foregoing expression for ds gives

$$ds = \left\{1 + 8n_1\left(1 - 2\frac{x}{a}\right)\left[2n_1\left(1 - 2\frac{x}{a}\right) + \sin\theta\right]\right\}^{\frac{1}{2}} \sec\theta\,dx.$$

Now this equation is in the form $ds = (1 + X)^{\frac{1}{2}} \sec\theta\,dx$, where

$$X = 8n_1(1 - 2x/a)\,[2n_1(1 - 2x/a) + \sin\theta].$$

Unless the sag is relatively large ds and $\sec\theta\,dx$ are nearly equal at all points along the curve (see Fig. 209); hence $(1 + X)$ is nearly equal to

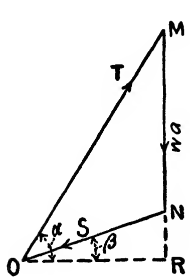

FIG. 210

[1] Let MN (Fig. 210) represent the load on the cable AB, and let MO and NO be parallel to the tangents at A and B (Fig. 209) respectively; then $OMNO$ is a force triangle for the three forces acting on the cable AB, and OM represents T and NO represents S. It is plain from the figure that $OR\tan\alpha - OR\tan\beta = MN$, or $OR(\tan\alpha - \tan\beta) = MN$. But $OR = T\cos\alpha = S\cos\beta$, and $MN = wa$; hence

$$T\cos\alpha\,(\tan\alpha - \tan\beta) = wa = S\cos\beta\,(\tan\alpha - \tan\beta).$$

Substituting the values of $\tan\alpha$ and $\tan\beta$ given by equation (6), it is found that

$$T\cos\alpha = wa^2/8f_1 = S\cos\beta.$$

1 at all points, which means that X is small compared with 1. Therefore $(1 + X)$ may be expanded by the binomial theorem, and all terms except the first few dropped without serious error. Thus, as a close approximation, $ds = (1 + \frac{1}{2}X - \frac{1}{8}X^2) \sec\theta\, dx$,

and $$l_1 = \int_0^a (1 + \tfrac{1}{2}X - \tfrac{1}{8}X^2) \sec\theta\, dx.$$

Substituting for X and X^2 their values, and integrating gives

$$l_1 = a_1 (1 + \tfrac{8}{3}\cos^2\theta \cdot n_1^2 - \tfrac{32}{5} n_1^4). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the approximation made in the derivation of formula (10) is not permissible in a given case, then one might determine the exact length of the cable AB somewhat as follows when a, b, and f_1 are given: First locate the vertex O of the parabola of which the cable is a part from equation (7).

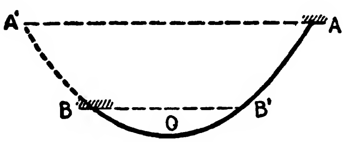

FIG. 211

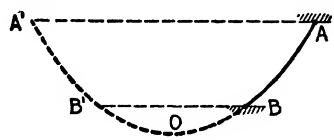

FIG. 212

The vertex will be found either between A and B, on the cable (Fig. 211), or beyond B (Fig. 212). Then determine the length of the arcs AOA' and BOB' by means of equation (5), Art. 95, and finally the length l_1 of the arc AB from

$l_1 = \frac{1}{2} AOA' + \frac{1}{2} BOB'$ for Fig. 211 or $l_1 = \frac{1}{2} AOA' - \frac{1}{2} BOB'$ for Fig. 212.

For example take $a = 800$ feet, $b = 300$ feet, and $f_1 = 200$ feet. Let x_0 and y_0 = the coördinates of the vertex. From equation (7)

$$x_0 = \frac{(300 + 4 \times 200)\,800}{8 \times 200} = 550, \text{ and } y_0 = \frac{(300 + 4 \times 200)^2}{16 \times 200} = 378.5.$$

Hence the cable hangs as shown in Fig. 213. The length $AA' = 1348.6$ feet according to (5), ($a = 1100$ and $n = 378.5 \div 1100$); the length $BB' = 530.9$ feet according to (5), ($a = 500$ and $n = 78.5 \div 500$). Hence $AB = \frac{1}{2} \times 1348.6 + \frac{1}{2} \times 530.9 = 939.8$ feet.

97. Catenary Cable; Symmetrical Case. — A chain or flexible cable suspended from two points and hanging freely under its own weight or a load uniformly distributed along its length assumes a curve called (common) catenary. Let A and B (Fig. 214) be the points of suspension of such a cable, C its lowest point, Q any other point of the cable, s = the length CQ, H = tension at C, T = tension at Q, ϕ = slope of the curve at Q, w = weight of load per unit length of cable, and c = a length so that $cw = H$ or $c = H/w$. The forces acting on CQ are H, T, and ws. Since they are in equilibrium, $T \cos\phi = H$, and $T \sin\phi = ws$; hence $\tan\phi = ws/H = s/c$. But $\tan\phi = dy/dx$, therefore

$$dy/dx = s/c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now since $ds^2 = dx^2 + dy^2$, $(ds/dy)^2 = (dx/dy)^2 + 1$ and $(ds/dx)^2 = 1 + (dy/dx)^2$; also

$$\left(\frac{ds}{dy}\right)^2 = \frac{c^2}{s^2} + 1 = \frac{c^2 + s^2}{s^2} \quad \text{and} \quad \left(\frac{ds}{dx}\right)^2 = 1 + \frac{s^2}{c^2} = \frac{c^2 + s^2}{c^2}. \quad . \ (2)$$

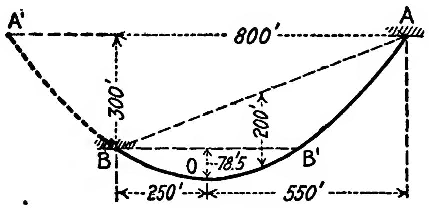

FIG. 213

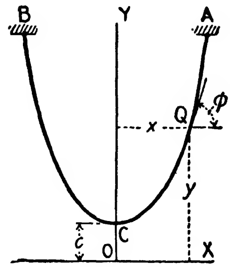

FIG. 214

Integrating the first one of these equations gives $y = (c^2 + s^2)^{\frac{1}{2}} + A$ where A is a constant of integration. But $y = c$ where $s = 0$, therefore $A = 0$, and hence

$$y^2 = c^2 + s^2, \quad \text{or} \quad s^2 = y^2 - c^2. \quad . \ . \ . \ . \ . \ . \ . \ . \ (3)$$

Integrating the second differential equation gives

$$x = c \log_e\left[\frac{s}{c} \pm \sqrt{\left(\frac{s}{c}\right)^2 + 1}\right] = c \sinh^{-1}\frac{s}{c}, \quad . \ . \ . \ . \ . \ (4)$$

the constant of integration being zero ($x = 0$ when $s = 0$). From (3)

$$s = \tfrac{1}{2} c\, (e^{x/c} - e^{-x/c}) = c \sinh\frac{x}{c}. \quad . \ . \ . \ . \ . \ . \ . \ . \ (5)$$

To obtain the cartesian equation of the catenary, combine (3) and (4) or (3) and (5) so as to eliminate s. Thus squaring (5) and comparing with (3) it is found that

$$y = \tfrac{1}{2} c\, (e^{x/c} + e^{-x/c}) = c \cosh\frac{x}{c}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (6)$$

or

$$x = c \log_e\left[\frac{y}{c} \pm \sqrt{\left(\frac{y}{c}\right)^2 - 1}\right] = c \cosh^{-1}\frac{y}{c}. \quad . \ . \ . \ . \ . \ (7)$$

The slope angle ϕ at any point in terms of the coördinates of the point (x,y,s) is given by

$$\tan\phi = s/c = \tfrac{1}{2}\,(e^{x/c} - e^{-x/c}) = \sinh\,(x/c). \quad . \ . \ . \ . \ . \ (8)$$

See equations (1) and (5). And, from equations (2) and (3),

$$\sin\phi = s/y \quad \text{and} \quad \cos\phi = c/y. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (9)$$

It follows from the equilibrium equation $T \sin\phi = ws$ and (9), that

$$T = wy, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (10)$$

that is, the tension at any point Q equals the weight of a length of cable reaching from Q to the directrix OX. Hence T increases from C to A. According to the definition of c

$$H = wc. \qquad (11)$$

In passing, it may be noted that since $T \cos \phi = H$, the horizontal component of the tension at any point $Q = wc$, constant for a given suspended cable.

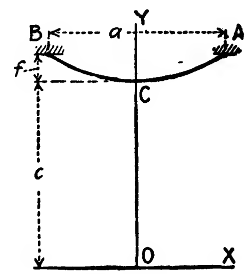

FIG. 215

As in the preceding article, let a = span AB (Fig. 215), f = sag, and l = length of cable ACB. Any two of the three dimensions a, l and f determine the catenary, as will be shown presently. For the point A, $x = \frac{1}{2}a$, $y = f + c$, and $s = \frac{1}{2}l$. Hence substituting in equations (3), (4) and (6) respectively gives

$$(f + c)^2 = c^2 + \tfrac{1}{4}l^2, \quad \text{or} \quad c/f = \tfrac{1}{8}(l/f)^2 - \tfrac{1}{2}. \qquad (3')$$

$$\tfrac{1}{2}a = c \sinh^{-1}(\tfrac{1}{2}l/c), \quad \text{or} \quad \tfrac{1}{2}a/c = \sinh^{-1}(\tfrac{1}{2}l/c). \qquad (4')$$

and $$f + c = c \cosh(\tfrac{1}{2}a/c), \quad \text{or} \quad 1 + (f/c) = \cosh(\tfrac{1}{2}a/c). \qquad (6')$$

When l and f are given (3′) gives c, and then a may be gotten from (4′) or (6′). When a and f are given (6′) determines c but the equation cannot be solved directly, — only by trial or by some similar method; having thus determined c, l may be gotten from (3′) or (4′). When a and l are given, (4′) determines c (solution by trial), and then f may be gotten from (3′) or (6′).

Inasmuch as these trial methods are generally long, computations on some catenary problems may be facilitated by means of diagrams. In Fig. 216 the curves marked A give the relation between f/a and l/a for values of f/a from 0 to 0.5 and (corresponding) values of l/a from 1 to about 1.50. For example, let $a = 800$ feet and $f = 160$ feet. Then $f/a = 0.20$, and the corresponding ordinate (over $f/a = 0.20$) to curve A reads 1.10; hence $l/a = 1.10$, and $l = 800 \times 1.10 = 880$ feet (length of cable).

Most practical catenary problems involve the strength of the wire or cable and the load per unit length of wire. For such problems one has, in addition to (3′), (4′) and (6′),

$$T = w(f + c), \quad \text{or} \quad T/w = f + c, \qquad (11')$$

where T = the greatest tension (at the points of support), which should of course not exceed the strength of the wire. Most of these problems can be solved by trial only, unless a diagram is available. For example, given the strength T of a wire, the load per unit length w, and the span a; required the proper length of wire l. Here

$$T/wa = f/a + c/a. \qquad (11'')$$

This equation and (6′) contain only two unknown quantities f and c, and the two equations determine f and c. But they can be solved only by trial. After f and c have been ascertained, then l may be computed from (3′) directly. The curves marked B in Fig. 216 show the relation between f/a and T/wa. Thus if $f/a = 0.20$, as in the preceding illustration, then the corresponding ordinate to curve B (over $f/a = 0.20$) reads 0.85; hence $T/wa = 0.85$ and $T = 0.85\,wa$.[1]

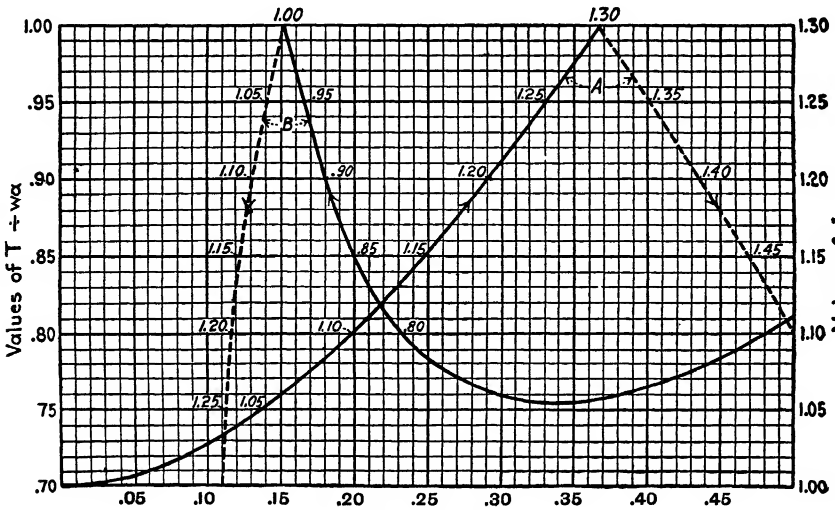

FIG. 216

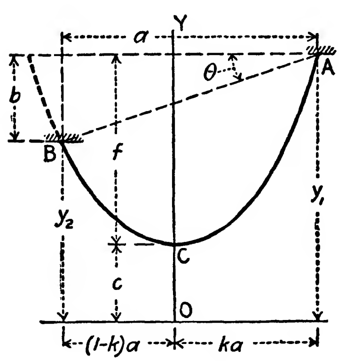

FIG. 217

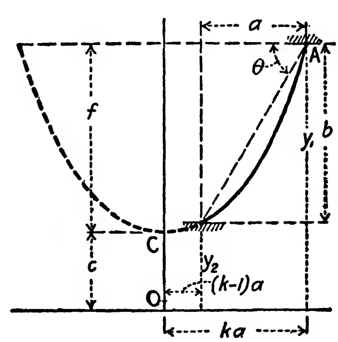

FIG. 218

98. Catenary Cable; Unsymmetrical Case. — The cable uniformly loaded along its length hangs in an arc of a catenary. The vertex C may be on the cable (between the points of suspension A and B) as in Fig. 217,

[1] Figure 216 was prepared from plate II of Mr. Thomas' paper mentioned in the footnote at the end of this chapter. (For cases of relatively small sag ratios, see that plate.)

or beyond the lower point of suspension as in Fig. 218. In either figure, a = the horizontal distance between A and B, b = the vertical distance, θ = angle between AB and the horizontal, f = sag or vertical distance AC, and l = arc AB. Most problems in this case as in the symmetrical case can be solved only by a trial method; hence diagrams are practically necessary in this case also.

In Fig. 219 there are two groups of curves relating to this unsymmetrical case; the group occupying the lower right-hand portion consists of graphs

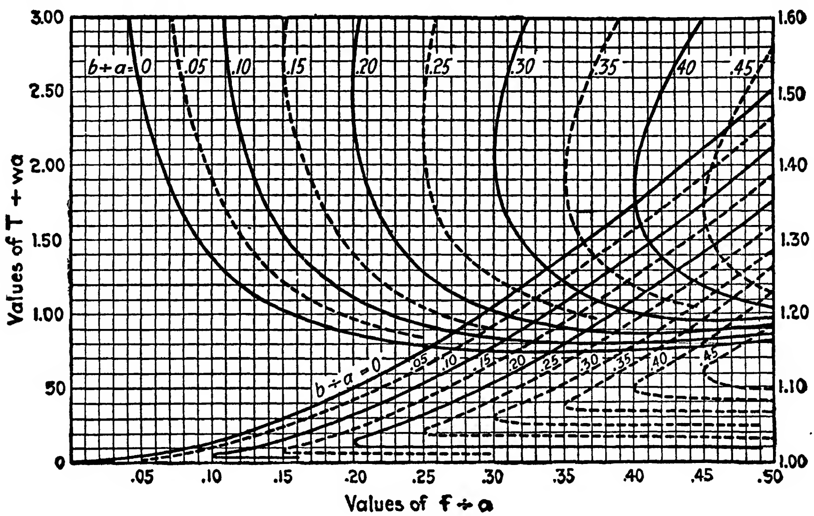

FIG. 219

showing the relation between f/a and l/a (values at right-hand margin) for ten values of b/a (slope of AB, Figs. 217 and 218). The other group consists of graphs showing the relation between f/a and T/wa (values at left-hand margin) for the same ten slopes (T = tension at higher point of support and w = weight of cable per unit length). To illustrate, let a = 200 feet, b = 40 feet, l = 240 feet, and w = 2 pounds per foot. On the curve for $b/a = 0.20$ in the lower group, find the point whose ordinate $l/a = 1.20$ and note that the abscissa of that point is $f/a = 0.385$. Hence $f = 200 \times 0.385 = 77$ feet. On the curve for $b/a = 0.20$ of the upper group, find the point whose abscissa is 0.385 and note that its ordinate $T/wa = 0.90$. Hence $T = 2 \times 200 \times 0.90 = 360$ pounds.[1]

99. Approximate Solutions of Catenary Problems. — If the cable is suspended from two points at the same level and the sag is small compared with the span so that the slope of the catenary is small at every point, then the load (weight) per unit length of span is nearly constant and equal to the weight of the cable per unit length. Hence the catenary coincides very

[1] Figure 219 was made from certain of the (more extensive) figures in Mr. Robertson's paper mentioned in the footnote at the end of this chapter.

nearly with a parabola of the given span and sag, and the formulas and results of Art. 95 may be applied to the case here under consideration without serious error.

That the catenary agrees closely with a parabola can be shown otherwise as follows: Expanding the exponentials in the equation of the catenary, (6) of Art. 97, gives

$$e^{x/c} = 1 + \frac{x}{c} + \frac{x^2}{2\,c^2} + \frac{x^3}{3\,c^3} + \cdots \text{ and } e^{-x/c} = 1 - \frac{x}{c} + \frac{x^2}{2\,c^2} - \frac{x^3}{3\,c^3} + \cdots;$$

hence the equation of the catenary may be written

$$y = \frac{c}{2}\left(2 + \frac{x^2}{c^2} + \cdots\right).$$

Neglecting the higher powers of the small quantity $\frac{x}{c}$ gives, as close approximations,

$$y = c + x^2/2\,c, \quad \text{or} \quad x^2 = 2\,c\,(y - c).$$

These are equations of a parabola whose axis coincides with the y coördinate axis and vertex c distant above the origin of coördinates.

If the supports A and B are not at the same level (Fig. 209) and the sag f of the cable is small compared to the distance between the points of support, then the slope of the catenary is nearly constant and the load per unit length of horizontal distance is nearly constant ($w \sec\theta$, where w = weight of cable per unit length, and θ = angle BAX). Hence the catenary coincides very nearly with a parabolic arc of the given (oblique) chord AB and sag f_1, and the formulas of Art. 96 may be applied to the cable under consideration without serious error, it being understood that w of that article = (weight of cable per unit length) $\times \sec\theta$.

100. Cables with Concentrated Loads; Cable Weight Negligible. — Let Fig. 220 represent a cable ACB suspended from two given points A

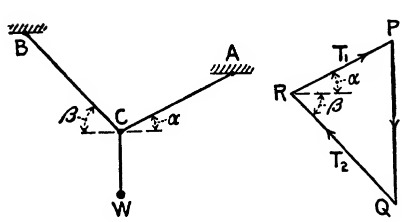

FIG. 220

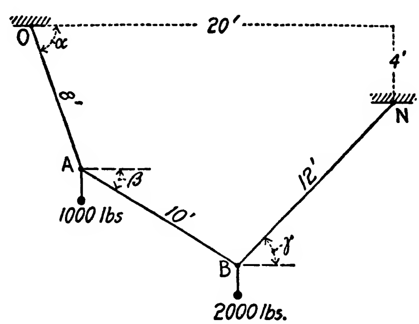

FIG. 221

and B, C being a given point from which a load is suspended. If the cable can be "laid out" in a drawing, the tensions in AC and BC can be determined easily by constructing the force triangle $PQRP$ for the load W and

the two tensions. $PQ = W$ according to some convenient scale; PR and QR (parallel to AC and BC respectively) represent the tensions in AC and BC. Or, if one wishes to avoid graphical methods, the two tensions (T_1 and T_2) may be computed by solving the triangle algebraically. Such solution would give

$$T_1 = W \cos\beta/\sin(\alpha + \beta) \quad \text{and} \quad T_2 = W \cos\alpha/\sin(\alpha + \beta),$$

where α and β are the angles which AC and BC make with the horizontal (Fig. 220).

When several bodies are suspended from given points on the cable, the cable takes up a definite position, but it is not easy to determine the slopes of the segments of the cable and the tensions. The difficulty lies in the algebraic computation. For example, consider the case represented in Fig. 221. The given data are shown in the figure; the lengths are drawn to scale, but the inclinations of the segments of the cable may not be correct, being unknown as yet. Let the inclinations be called α, β and γ as shown; and T_1, T_2 and T_3 = the tensions in OA, AB and BN respectively. At each point of suspension of a load (A or B) there are three forces acting; at A, the load 1000 pounds, T_1 and T_2, and at B, the load 2000 pounds, T_2 and T_3. Consideration of forces at A and of those at B gives respectively

$$T_1 \cos\alpha = T_2 \cos\beta \quad \text{and} \quad T_1 \sin\alpha - T_2 \sin\beta = 1000$$
$$T_2 \cos\beta = T_3 \cos\gamma \quad \text{and} \quad T_2 \sin\beta + T_3 \sin\gamma = 2000.$$

It is plain from the geometry of the figure that

$$8\cos\alpha + 10\cos\beta + 12\cos\gamma = 20, \text{ and } 8\sin\alpha + 10\sin\beta - 12\sin\gamma = 4.$$

These six equations may be solved simultaneously for the six unknowns (T_1, T_2, T_3, α, β and γ); the actual solution is not simple. For similar cases with more than two loads, the work of solving the equations increases rapidly with increasing number of loads.

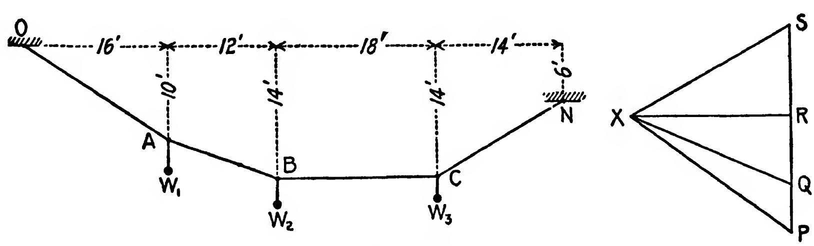

FIG. 222

Suspended loads can be chosen so as to hold points of suspension (A, B, etc.) in certain definite positions. For instance let it be required to determine W_1, W_2, etc., to hold a cable in the position shown in Fig. 222. One may assume any value for one of the weights and then compute the values of the others. Thus taking $W_1 = 1000$ pounds say, compute the

tension in AB from a force triangle for the three forces acting at A. $PQXP$ is such a triangle, where PQ = 1000 pounds (according to any convenient scale) and PX and QX are parallel to OA and AB respectively; then XQ represents the tension in AB. The next step is to find the value of W_2 which corresponds to such tension in AB; so draw a force triangle for the three forces acting at B one of which is the determined tension in AB. This force triangle is $QXRQ$, and so RQ represents W_2 and XR represents the tension in BC. Finally, draw the force triangle $RXSR$ for the three forces acting at C, one of which is the determined tension in BC, and thus find that W_3 is represented by SR. Obviously any three weights W_1, W_2 and W_3 in the proportion of PQ, QR and RS would hold the cable in the specified position.

101. Cables with Concentrated Loads; Cable Weight not Negligible. — It is assumed in the following discussion that the cable segments are nearly flat so that they are practically parabolic arcs (see Art. 99). Then the weight of any segment of the cable is practically the same as the weight of a length equal to the chord of the segment. Let ABC (Fig. 223) be a cable supported at A and C, a load being suspended from the cable at its middle point B. Given the span $AC = 2a$, the length of the cable $= 2l$, the weight of the cable per unit length $= w$, and the load $= W$; required the sag (depth of B below AC) and the tension at A. This (apparently) simple problem is determinate but practically unsolvable on account of algebraic difficulties. The equations are easily set up.

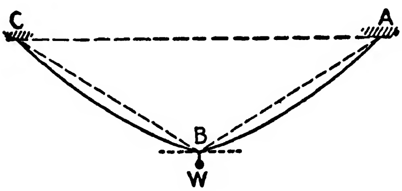

FIG. 223

Thus let a_1 = the (unknown) length of chord AB, f_1 = the sag of the cable below the chord as in Fig. 209, S = the tension and β = the slope of the cable at B (Fig. 209). Then according to equations (10) and (6) respectively of Art. 96,

$$(1) \qquad \frac{l}{a_1} = 1 + \frac{8}{3}\left(\frac{a}{a_1}\right)^2\left(\frac{f_1}{a_1}\right)^2 \quad \text{and} \quad \tan\beta = \frac{\sqrt{a_1^2 - a^2}}{a} - \frac{4f_1}{a}. \quad . \ . \ (2)$$

According to the footnote on page 118.

$$S\cos\beta = (wa_1/a)\,a^2/8f_1. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (3)$$

From the three forces acting at B (W, S and S), it is plain that

$$2S\sin\beta = W. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (4)$$

These four equations determine the unknowns appearing in them, a_1, f_1, S, and β. Thus by division, the last two give $\tan\beta = 4Wf_1/wa_1a$; equating the two values of $\tan\beta$ and transforming gives

$$\frac{a}{a_1}\frac{4W}{wa}\frac{f_1}{a_1} = \sqrt{1 - \left(\frac{a}{a_1}\right)^2} - 4\frac{f_1}{a_1}. \quad . \ . \ . \ . \ . \ . \ . \ (5)$$

This equation and (1) contain only two unknowns, the ratios (a/a_1) and (f_1/a_1), and the equations determine the ratios. Supposing the ratios determined one may find a_1 since a is given, and then f_1. Exact simultaneous solution of equations (1) and (5) is impossible, but each equation may be graphed and then the coördinates of their intersection would be the desired values of a/a_1 and f_1/a_1.

The converse of the preceding problem is much simpler. It is this: Given the span $AC = 2\,a$, the chord $AB = a_1$, the sag f_1, and the weight of the cable per unit length w; required the load W. Equations (2), (3) and (4) give in succession β, S and W. Equation (1) gives the length l.[1]

[1] For other information on the subjects of this chapter, particularly as related to electric transmission lines, see the following: University of Illinois Bulletin, No. 11 (1912), by A. Gruell; Transactions American Institute of Electric Engineers, Vol. 30 (1911), papers by Wm. L. Robertson, Percy H. Thomas, and Harold Pender and H. F. Thompson. These papers contain extensive tables and diagrams, and discuss effects of temperature changes.

KINEMATICS

CHAPTER VIII

MOTION OF A POINT

§ 1. Rectilinear Motion

102. Path; Position; Displacement. — The line along which a moving point travels is called the *path* of the point, or path of the motion. If the path is a straight line the point is said to have *rectilinear motion;* if the path is a curved line the point is said to have *curvilinear motion.* In the articles that immediately follow we discuss rectilinear motion only, and for convenience we shall assume that the path is horizontal unless otherwise stated.

The position, at any instant, of a point that has rectilinear motion is conveniently specified by its distance from some fixed origin in the path; this distance we call the *position abscissa* of the point and designate by s. For distinction s is considered positive when the point is on one side of the origin and negative when on the other; the rule as to sign is arbitrary, but once made should be adhered to throughout the discussion of any given case of motion. We shall here call s positive when measured to the right of the origin and negative when measured to the left of the origin, and in accordance with this rule we shall speak of the positive direction and the negative direction, meaning to the right and to the left respectively.[1]

The *displacement* of a point for any given interval of time is the distance from its position at the beginning of that interval to its position at the end of that interval. In accordance with the rule of sign stated above we consider displacement positive when measured to the right and negative when measured to the left. From the definition it is evident that the displacement of a point for any time interval is Δs, the increment[2] in its position abscissa for that interval.

[1] It must be noted that here, and subsequently, sign is used simply as a convenient means of distinguishing between two opposite directions, and has nothing to do with magnitude. A negative position abscissa (or other quantity) is not necessarily less than a positive one. But we speak of it as *algebraically* less, and say, accordingly, that such a quantity (if varying) increases or decreases, *algebraically.*

[2] By increment of a quantity is meant the change that quantity undergoes in some given interval of time; it is computed by subtracting the *initial* value of the quantity from the *final* value. Thus if at one instant there are 12 gallons of water in a vessel and at a later instant 10 gallons, the increment in the volume of water is $10 - 12 = -2$ gallons.

If the direction of motion does not change during a given interval of time, the displacement for that interval is equal to the distance actually traveled by the moving point during that interval. But if the motion is reversed, the distance traveled is greater than the displacement. Thus if a point starts at $s = +3$, moves out to $s = +10$, and then back to $s = +6$, the displacement (for the entire motion) is $+6 - 3 = +3$, but the distance traveled is 7 (out) + 4 (back) = 11.

103. Uniform and Nonuniform Motion. — When a point moves so that it undergoes equal displacements in all equal intervals of time it has *uniform motion;* if it moves in any other way it has *nonuniform motion.* To have uniform motion, then, a point must move continuously in the same direction with constant speed.

104. Velocity. — By *velocity* of a moving point is meant the time rate at which its displacement occurs. We denote velocity by v.

In any uniform motion the velocity is constant, and may be computed by dividing the displacement for any interval of time by the interval. Thus if v = velocity, Δs = displacement, and Δt = the interval of time, then

$$v = \frac{\Delta s}{\Delta t}. \qquad (1)$$

In any nonuniform motion the velocity is not constant but changes continuously. The above quotient is regarded as the *average velocity* for the interval Δt, that is

$$v_a = \frac{\Delta s}{\Delta t}, \qquad (2)$$

where v_a denotes average velocity. And the velocity at any particular instant is the limit of the average velocity for an interval including the instant in question as the interval is taken smaller and smaller. In the calculus notation this limit is ds/dt, hence

$$v = \frac{ds}{dt}. \qquad (3)$$

It is evident from the above equations that v_a and v are positive or negative according as the displacement Δs and the elementary displacement ds are positive or negative. Velocity, therefore, like displacement, is regarded as having sign, the sign being determined by the *direction* of the motion. In conformity with the (arbitrary) rule for sign given in Art. 102, we consider velocity positive when the point is moving to the right and negative when the point is moving to the left. (When a point has positive velocity, positive displacement is occurring and s is algebraically increasing; when a point has negative velocity, negative displacement is occurring and s is algebraically decreasing.)

Equations (1), (2) and (3) imply that the unit velocity is the velocity

of a point moving uniformly and describing unit distance in unit time; for example, the mile per hour (mi/hr.), foot per second (ft/sec.), etc.[1]

105. Calculation of Velocity. — If the relation between s and t for a particular motion is known so that it can be expressed by an equation the velocity can be found by means of the formula $v = ds/dt$; differentiation of the equation gives the formula for v. This method of determining the velocity is illustrated in Ex. 1, 2 and 3 below.

If the relation between s and t cannot be expressed by an equation, the above formula cannot be used directly to find the velocity. One can, however, ascertain the approximate value of v at any instant as follows: Using equation (2), compute the average velocity for each of a series of shorter and shorter intervals, all of which include the instant in question; then ascertain, as nearly as practicable, the limit approached by this average velocity as the interval approaches zero. In order to do this one must have a series of corresponding or simultaneous values of s and t over a period of time including the instant in question. This method of determining the velocity is illustrated in Ex. 4 below.

Graphical methods of determining velocity are discussed in Art. 111.

EXAMPLE 1. A point is known to move so that its distance (in feet) from the starting point always equals four times the square of the time (in minutes) after starting. It is required to determine the general formula for the velocity of the point, and to determine this velocity two minutes after starting and also when the point has traveled 100 ft.

Solution: From the description of the motion it is evident that

$$s = 4\,t^2. \quad \text{Therefore } v = ds/dt = 8\,t.$$

This is the general formula for the velocity. When $t = 2$, $v = 8 \times 2 = 16$ ft/min. When $s = 100$, $t = \sqrt{100 \div 4} = 5$ min. and $v = 8 \times 5 = 40$ ft/min.

EXAMPLE 2. Figure 224 represents a simple hoisting rig. A cable C runs over a pulley and is attached at one end to the load W and at the other to a truck or team B which, moving to the left, raises the load. Assuming the dimensions given on the figure, and assuming that the truck is run at a uniform velocity of 5 mi/hr., it is required to determine the general formula for the velocity of the load and the value of this velocity when the truck has moved 10 ft. (The diameter of the pulley may be considered zero and the cable may be assumed attached to the truck at the ground level.)

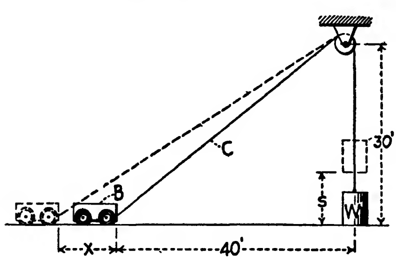

FIG. 224

Solution: Taking the origin at the initial position of the load, s will be the distance measured up along the vertical path to any subsequent position as shown. The first step in the solution is to express s in terms of t.

The velocity of the truck = 5 mi/hr. = 7.33 ft/sec.; therefore the distance it will have moved to the left at the end of t seconds is

$$x = 7.33\,t \text{ ft.}$$

[1] For dimensions of a unit velocity see Appendix A.

The corresponding horizontal distance from truck to pulley will be $40 + 7.33\,t$, the length of cable from truck to pulley will be $\sqrt{(7.33\,t + 40)^2 + 30^2}$, the length of cable from pulley to load will be the total length, 80 ft., minus this, or $80 - \sqrt{(7.33t + 40)^2 + 30^2}$ and the distance from the load to the ground will be 30 minus this, or

$$s = \sqrt{(7.33\,t + 40)^2 + 30^2} - 50.$$

Then
$$v = \frac{ds}{dt} = \frac{(7.33\,t + 40)\,(7.33)}{[(7.33\,t + 40)^2 + 30^2]^{\frac{1}{2}}}.$$

This is the general formula for the velocity of the load.

When the truck has moved 10 ft., $t = 10 \div 7.33 = 1.36$ sec., and substitution of this value of t in the equation for v gives $v = 6.28$ ft/sec.

EXAMPLE 3. Figure 225 represents a device known as the Scotch cross-head; when the crank rotates the piston is given a reciprocating motion which is connected in an obvious way with the motion of the crank pin P, the position of which is defined by the coördinates x and y. Denoting by r the length of the crank OP and by n the number of revolutions of the crank per unit time (assumed constant), it is required to develop the general formula for the velocity of the piston, and to determine this velocity for the assumed data $r = 9$ in., $n = 150$ rev/min., and for the two positions of the piston corresponding to (i) $x = 3$ in., y positive. and (ii) $x = 3$ in., y negative. Counter-clockwise rotation will be assumed.

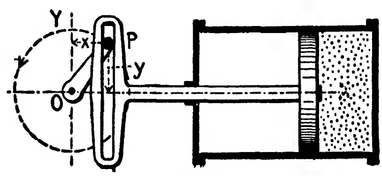

FIG. 225

Solution: It is first necessary to choose an origin from which to measure s and an instant from which to date t. It will be convenient to take the origin at the mid-position of the piston and to consider $t = 0$ when the piston is farthest to the right. Next, s must be expressed in terms of t. Let θ = the angle measured from the horizontal to the crank. In unit time the crank turns through n revolutions or $2\,\pi n$ radians, therefore after time t, $\theta = 2\,\pi nt$ rad. Obviously $s = x$ and $x = r \cos \theta$, hence

$$s = r \cos 2\,\pi nt.$$

Differentiating with respect to t it is found that

$$v = -2\,\pi nr \sin 2\,\pi nt.$$

This is the general formula for the velocity of the piston in terms of t. By substituting for $\sin 2\,\pi nt$ its equivalent $\sqrt{1 - (\cos 2\,\pi nt)^2}$ and for $\cos 2\,\pi nt$ its value s/r, there is obtained

$$v = -2\,\pi nr \sqrt{1 - (s/r)^2}.$$

This is the general formula for the velocity of the piston in terms of s.

For $r = 9$ in. and $n = 150$ rev/min., the first of the above formulas becomes $v = -45\,\pi \sin 5\,\pi t$, where t = time in seconds and v = velocity in in/sec. For the position specified in (i) above, $s = 3$ in., $\theta = \cos^{-1} 3/9 = 70°\,30'$, and $t = 0.0782$ sec. Substitution of this value of t in the preceding equation for v gives $v = -133$ in/sec. The negative sign shows that the direction of the motion is toward the left. For the position specified in (ii) above, $\theta = 289°\,30'$ and $t = 0.322$ sec. Substitution of this value of t gives $v = 133$ in/sec. The positive sign shows that the direction of the motion is toward the right.

Solution might also have been effected by use of the second general formula. For $r = 9$ in. and $n = 150$ rev/min., this becomes $v = -45\,\pi \sqrt{1 - (s/9)^2}$, and substitution of $s = 3$ gives $v = \pm 133$ in/sec.

(The motion of the piston described in this example is what is known as Simple Harmonic Motion, an important special motion which is discussed at length in Art. 120.

Pistons which operate or are operated by an ordinary connecting rod do not have simple harmonic motion, though, as shown in Art. 123, they have a somewhat similar one.)

EXAMPLE 4. In a certain launching the ship moved through the distances given after s in the times given after t in the schedule below.

$t =$	0	2	4	6	8	10	12	14	16 sec.
$s =$	0	3.4	9.3	17.3	27.4	39.6	53.4	69.4	88.0 ft.

It is required to determine the velocity of the ship eight seconds after the motion commenced.

Solution: The average velocity is computed for each of a series of shorter and shorter intervals, each including the instant $t = 8$ secs. It is convenient to take intervals commencing with the instant in question, as 8 to 16, 8 to 14, etc. The computations are tabulated in the schedule below, time intervals being given under Δt, corresponding displacements under Δs and corresponding average velocities under v_a.

Δt (secs.)	Δs (ft.)	v_a (ft/sec.)
8 to 16 = 8	60.6	7.57
8 to 14 = 6	42.0	7.00
8 to 12 = 4	26.0	6.50
8 to 10 = 2	12.2	6.10

It is apparent that for the intervals considered the average velocity decreases as the interval becomes shorter. It is reasonable to suppose that the average velocity for the intervals 8 to 9, 8 to $8\frac{1}{2}$, 8 to $8\frac{1}{4}$, etc., continues to decrease as the interval of time approaches zero. In thus decreasing it approaches a definite limit, and the column of average velocities suggests that this definite limit might be about 5.8 ft. per sec. The exact value of the limit is the rate at which the ship was moving at the time 8 secs.

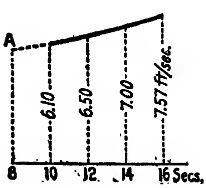

FIG. 226

The limit of the average velocities may be approximately ascertained by the graphical construction shown in Fig. 226. The average velocities for the intervals considered are plotted in a manner which is obvious, and a smooth curve is drawn through the plotted points. This curve extended (with the same general trend of curvature) intercepts the vertical through point 8 at A, and the ordinate 8–A represents approximately the limit sought, that is, the velocity at the instant $t = 8$ sec.

106. Speed. — It is at times desirable to speak of the rate of motion without regard to direction, that is of the magnitude of velocity without regard to sign. We use the word *speed* in this sense, — to express, simply, *how fast* a point moves. If two trains are running at 60 miles per hour, one toward the east, the other toward the west, they have the same speed but different velocities.

The *average speed* for a given interval of time is the *distance* traveled during that interval divided by the interval. It should be noted that the average speed is equal to the magnitude of the average velocity for any interval during which the direction of the motion is not reversed, but greater for any interval during which the direction of motion is reversed. This follows from the fact that when the motion is in one direc-

tion, the distance traveled is equal to the displacement, and when the motion is reversed the distance traveled is greater than the displacement (Art. 102).

107. Acceleration. — By acceleration of a moving point is meant the time rate at which its velocity changes. We denote acceleration by a.

It has been seen that if a point has uniform motion its velocity is constant; a point so moving therefore has no acceleration. But if a point has nonuniform motion its velocity changes, and the point has acceleration.

If the velocity changes uniformly (equal velocity-increments in all equal intervals of time), then the acceleration is constant and may be computed by dividing the velocity-increment for any interval of time by the interval. Thus if a = acceleration, Δv = velocity-increment, and Δt = the interval of time, then

$$a = \frac{\Delta v}{\Delta t} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the velocity does not change uniformly, then the acceleration is not constant but changes continuously. The above quotient is regarded as the *average acceleration* for the interval t, that is

$$a_a = \frac{\Delta v}{\Delta t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a_a denotes average acceleration. And the acceleration at any particular instant is the limit of the average acceleration for an interval including the instant as the interval is taken smaller and smaller. In the calculus notation, this limit is dv/dt, hence

$$a = dv/dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If for v its value ds/dt (Eq. 3, Art. 104) be substituted, it is seen that $a = d^2s/dt^2$.

It is evident from the above equations that a_a and a are positive or negative according as the velocity-increment Δv and the elementary velocity increment dv, respectively, are positive or negative. Acceleration, therefore, like displacement and velocity, is regarded as having sign. Acceleration is positive when positive velocity is being taken on; negative when negative velocity is being taken on. And so acceleration has direction, — the direction of the velocity that is being taken on. It should be particularly noted that the sign of the acceleration does not depend merely on whether the *speed* is increasing or decreasing. Let toward the right be considered the positive direction. Then if a point is moving toward the right and going faster and faster, it has positive acceleration; if it is moving toward the right and going slower and slower, it has negative acceleration; if it is moving toward the left and going faster and faster, it has negative acceleration; if it is moving toward the left and going slower and slower, it has positive acceleration.

Equations (1), (2) and (3) imply that the unit acceleration is the acceleration of a point whose velocity changes uniformly by one unit (of velocity) in one unit of time, for example, the foot-per-second per second (ft/sec/sec.), mile-per-hour per minute (mi/hr/min.), etc.[1]

108. Calculation of Acceleration. — If the relation between v and t for a particular motion is known so that it can be expressed by an equation (or, as we say, the v-t law is known), the acceleration can be found by means of the formula $a = dv/dt$; differentiation of the equation gives the formula for a. This method of determining the acceleration is illustrated in Ex. 1 and 2 below.

If the relation between v and t cannot be expressed by an equation, the above formula cannot be used directly to find the acceleration. One can, however, ascertain the approximate value of v at any instant as follows: Using the formula $a_a = \Delta v/\Delta t$, compute the average acceleration for each of a series of shorter and shorter intervals, all of which include the instant in question; then ascertain, as nearly as practicable, the limit approached by this average as the interval approaches zero. In order to do this one must have a series of corresponding or simultaneous values of v and t over a period of time including the instant in question. This method of determining the acceleration is illustrated in Ex. 3 below.

Graphical methods of determining acceleration are discussed in Art. 112 and 114.

EXAMPLE 1. A point is known to move in a straight line so that the velocity (in miles per hour) always equals one-tenth of the square of the time (in seconds) after starting. It is required to develop the formula for the acceleration of the point and to determine the acceleration 3 sec. after the point starts.

Solution: From the description of the motion it is evident that

$$v = 0.1\,t^2.$$

Therefore

$$a = dv/dt = 0.2\,t.$$

This is the general formula for the acceleration of the point. When $t = 3$, $a = 0.2 \times 3 = 0.6$ mi/hr/sec.

EXAMPLE 2. It is required to develop the general formula for the acceleration of the piston of Ex. 3, Art. 105, and to determine the acceleration when the piston is in each of the positions specified in that example.

Solution: It was found that the velocity of the piston was given by

$$v = -2\,\pi nr \sin 2\,\pi nt$$

where t = time dated from the instant the piston was farthest to the right. Differentiating with respect to t it is found that

$$a = -4\,\pi^2 n^2 r \cos 2\,\pi nt.$$

This is the general formula for the acceleration of the piston in terms of t. By substituting for $\cos 2\,\pi nt$ its value s/r, there is obtained

$$a = -4\,\pi^2 n^2 s.$$

This is the general formula for the acceleration of the piston in terms of s.

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[1] For dimensions of a unit acceleration, see Appendix A.

For $r = 9$ in. and $n = 150$ rev/min., the second of the above formulas becomes $a = -25\,\pi^2 s$, where s = position abscissa in inches and a = acceleration in inches per second per second. For both the positions specified ($x = 3$ in., y positive, and $x = 3$ in., y negative) $s = 3$, and substitution gives $a = -740$ in/sec/sec.

EXAMPLE 3. In a certain starting test of an electric street railway car, corresponding values of velocity and time after starting were as given in the schedule below.

$t = 0$	1	2	3	4	5	6	7	8	9	10	sec.;
$v = 0$	2.8	5.3	7.7	9.9	11.9	13.7	15.2	16.4	17.3	18.0	mi. per hr.

It is required to determine the (approximate) acceleration of the car 6 sec. after starting.

Solution: The average acceleration is computed for each of a series of shorter and shorter intervals, each including the instant $t = 6$ sec. It is convenient to take intervals terminating with the instant in question, as 0 to 6, 1 to 6, etc. The computations are tabulated in the schedule below, time intervals being given under Δt, corresponding velocity increments under Δv, and corresponding average accelerations under a_a.

Δt (secs.)	Δv (mi/hr.)	a_a (mi/hr/sec.)
0 to 6 = 6	13.7	2.28
1 to 6 = 5	10.9	2.18
2 to 6 = 4	8.4	2.10
3 to 6 = 3	6.0	2.00
4 to 6 = 2	3.8	1.90
5 to 6 = 1	1.8	1.80

It is apparent that for the intervals considered the average acceleration decreases as the interval becomes shorter. It is reasonable to suppose that the average acceleration for the intervals $5\frac{1}{2}$ to 6, $5\frac{3}{4}$ to 6, etc., continues to decrease as the interval of time approaches zero. In thus decreasing it approaches a definite limit, and the column of average accelerations suggests that this definite limit might be about 1.7 mi/hr/sec. The exact value of the limit is the acceleration 6 sec. after the car started.

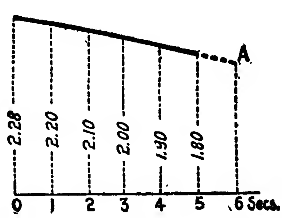


FIG. 227

The limit of the average accelerations may be approximately ascertained by the graphical construction shown in Fig. 227. The average accelerations for the intervals considered are plotted in a manner which is obvious, and a smooth curve is drawn through the plotted points. This curve extended intercepts the vertical through point 6 at A, and the ordinate 6–A represents the limit sought, — that is, the acceleration at the instant $t = 6$ sec.

109. Features of a Motion Determined by Integration.

— In the preceding articles we established the relations

$$v = ds/dt, \quad \text{and} \quad a = dv/dt$$

and showed how, by differentiation, the v-t law could be determined from the s-t law and the a-t law from the v-t law. By the reverse process, integration, one can determine the v-t law from the a-t law and the s-t law from the v-t law, using the relations

$$ds = v\,dt, \quad \text{and} \quad dv = a\,dt.$$

Integration leads to formulas for s and v (see Ex. 1 and 2 below).

We have so far considered only cases in which the motion is defined by the s-t, v-t or a-t laws. It may be defined otherwise — as by the a-s, the v-s or the a-v laws. If the a-s law for a motion is given, then the v-s law can be found by integrating the equation, $vdv = ads$, which equation is derived as follows:

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v, \text{ whence } vdv = ads.$$

If the a-v law is known, one can determine the time interval required for a given change in velocity to occur. For since $a = dv/dt$, $dt = dv/a$, and if v_1 is the velocity at time t_1, and v_2 is the velocity at time t_2, the interval $t_2 - t_1$ required for the velocity to change from v_1 to v_2 is found by integration as follows:

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{v_1}^{v_2} \frac{1}{a} dv.$$

If the v-s law is known, one can determine the time interval required for a given displacement. For since $v = ds/dt$, $dt = ds/v$, and so if s_1 is the position abscissa at time t_1 and s_2 is the position abscissa at time t_2, the interval $t_2 - t_1$ required for the displacement $s_2 - s_1$ to occur is found by integrating as follows:

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{s_1}^{s_2} \frac{1}{v} ds.$$

EXAMPLE 1. A point starts from the origin and moves according to the law $v = 60\,t + 4$, where v = velocity in feet per second and t = time in seconds after starting. It is required to determine the s-t law for the motion and to determine the distance traveled by the point during the interval $t = 3$ to $t = 6$.

Solution: Since $v = 60\,t + 4$, then $ds = (60\,t + 4)dt$, and integration gives $s = 30\,t^2 + 4\,t + C$. Since the point starts from the origin, $s = 0$ when $t = 0$. Substitution of these simultaneous values of s and t in the equation shows that $C = 0$. Therefore the s-t law for the motion is

$$s = 30\,t^2 + 4\,t.$$

Substitution in this equation shows that when $t = 3$, $s = 282$ ft., and when $t = 6$, $s = 1104$ ft. Therefore for the interval $t = 3$ to $t = 6$, $\Delta s = 1104 - 282 = 822$ ft. It is evident that v is positive for all values of t between 3 and 6, therefore the direction of the motion does not reverse during this interval, therefore the distance traveled $= \Delta s =$ 822 ft.

Solution of both parts of the problem might have been effected by integrating between limits. Thus, for the s-t law,

$$\int_0^s ds = \int_0^t (60\,t + 4)dt, \text{ whence } s = 30\,t^2 + 4\,t,$$

and to determine the distance traveled during the given interval

$$\int_{s_1}^{s_2} ds = \int_3^6 (60\,t + 4)dt, \text{ whence } s_2 - s_1 = \Delta s = 822 \text{ ft.}$$

EXAMPLE 2. A point moves according to the law $a = \cos t$, where a = acceleration in feet per second per second and t = time in seconds. The initial conditions are that

$v = 4$ when $t = 0$. It is required to determine the v-t law for the motion and to determine the velocity when $t = 3$ sec.

Solution: Since $a = \cos t$, $dv = (\cos t)dt$, and integration gives $v = \sin t + C$.

Substitution of the (initial) simultaneous values of v and t gives $4 = \sin 0 + C$, whence $C = 4$. Therefore the v-t law for the motion is $v = \sin t + 4$.

When $t = 3$, $v = \sin(3 \text{ radians}) + 4 = 4.14$ ft/sec.

Solution of both parts of the problem might have been effected by integrating between limits. Thus, for the v-t law

$$\int_4^v dv = \int_0^t \cos t\, dt,$$

$$\text{whence} \quad v - 4 = \sin t, \quad \text{or} \quad v = \sin t + 4;$$

and to determine the velocity when $t = 3$ sec.

$$\int_4^v dv = \int_0^3 \cos t\, dt,$$

$$\text{whence} \quad v - 4 = 0.14, \quad \text{or} \quad v = 4.14.$$

110. Motion Graphs. — It has been seen that a rectilinear motion may be described by an equation which expresses the mathematical relation between any two of the quantities s, v, a and t. It may also be described by a curve, representing the given equation or drawn through points plotted on rectangular axes to correspond to simultaneous values of any two of these quantities. Such a curve is called a *motion graph.* Motion graphs are of great aid in the study of a motion; they picture the events of the motion, so to speak; they show nicely the relations between displacement, velocity, acceleration and time, and they can be employed to advantage in the graphical solution of many problems. In the following articles we discuss a number of the more important motion graphs and illustrate their application.

111. Space-time Graph. — The space-time (s-t) graph is a curve constructed with t as abscissa and s as ordinate. It is sometimes called a distance-time graph; we avoid this because we usually mean by *distance* the travel of the point, and it is not this, but the position abscissa s, that is plotted. We might more accurately call the curve an abscissa-time graph, but the term space-time is in general use.

Since the slope of the s-t graph is proportional to ds/dt, and $v = ds/dt$ the slope at any point of the graph represents the velocity at the corresponding instant, according to some scale. This scale depends on the scales used in plotting the s-t graph. The slope of the curve is determined by drawing a tangent at the point in question.[1]

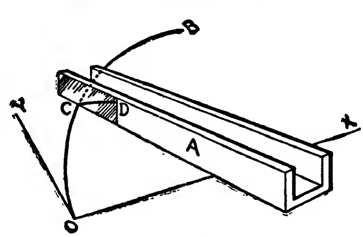


FIG. 228

[1] Several instruments have been devised recently for drawing a tangent to a plane curve. A very simple one is represented in Fig. 228. It consists of a metal straight-edge A with a portion of one side polished to a mirror. OB represents a curve on a piece of

EXAMPLE. The schedule of simultaneous values of s and t, describing the motion of the launched ship of Ex. 4, Art. 105, is repeated below.

$t = 0$	2	4	6	8	10	12	14	16	seconds
$s = 0$	3.4	9.3	17.3	27.4	39.6	53.4	69.4	88.0	feet.

It is required to determine, by graphical methods, the velocity of the ship 8 sec. after motion commenced.

Solution: The s-t graph (Fig. 229) for the entire motion is constructed; the scales chosen being 1 in. of ordinate = 100 ft. and 1 in. of abscissa = 10 sec., the slope scale is unit slope = $100 \div 10 = 10$ ft/sec.

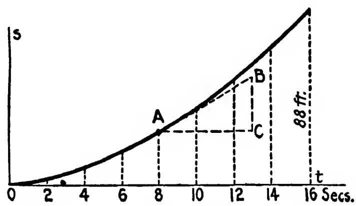


FIG. 229

A tangent line is drawn at A (point on the graph corresponding to $t = 8$ sec.) and from any point B on this tangent a perpendicular is dropped to the horizontal through A. Then AC and CB are measured, the slope computed, and the velocity determined by application of the slope scale, thus:

$AC = 0.5$ in., $CB = 0.27$ in., slope $= CB \div AC = 0.54$, $v = 10 \times 0.54 = 5.4$ ft/sec.

(Obviously the computation of the slope is somewhat simplified if AC is made an even number of inches.)

Instead of employing the slope scale as above one might measure AC and CB directly in terms of time and distance respectively; then the ratio $CB \div AC$ (as measured) = velocity. Thus:

$$AC = 5 \text{ sec.}, CB = 27 \text{ ft.}, v = 27 \div 5 = 5.4 \text{ ft/sec.}$$

112. Velocity-time graph. — The velocity-time (v-t) graph is a curve constructed with t as abscissa and v as ordinate. Since the slope of the v-t graph is proportional to dv/dt and $a = dv/dt$, the slope at any point of the graph represents the acceleration at the corresponding instant, according to some scale. This scale depends on the scales used in plotting the v-t graph. This method of determining velocity is illustrated in Ex. 1 below.

Since the area under the v-t graph (between the curve, the time axis and ordinates at t_1 and t_2) is given by $\int_{t_1}^{t_2} vdt$, and since $\int_{t_1}^{t_2} vdt$ = displace-

paper across which the straight-edge is laid at random but so that a portion of the curve is reflected from the mirror. The image CD and the curve CO are not smoothly continuous; there is a cusp at C. But if the instrument be turned about C until the cusp disappears, the curve merging smoothly into its image, then the straight-edge A is normal to the curve OB at C. Having located the normal at C, it is easy to draw the tangent. The principle of this instrument is the basis of Wagener's derivator (see Gramberg's *Technische Messungen*) by means of which the slope of a curve at any point can be read directly, without drawing the tangent or normal. An autographic form of (mirror) derivator has been devised by A. Elmendorf (see *Sci. Am. Suppl.* for Feb. 12, 1916).

Guillery's "aphegraphe" is another instrument for drawing a tangent to a plane curve. A metal strip or batten must first be fitted to the curve before the instrument proper can be applied. For full description of the aphegraphe, see *Mem. Soc. Ing. Civ. de France*, Bull. for April, 1911, where M. Guillery also explains how he applied his instrument to determine the acceleration-time curves for several mechanisms, and in particular the a-t curve for the "tup" of an impact testing machine during a blow.

ment for the interval t_1 to t_2, the area under the curve represents displacement according to some scale. This scale depends upon the scales used in plotting the v-t graph. This method of determining displacement is illustrated in Ex. 2 below.

EXAMPLE 1. The schedule of simultaneous values of v and t, describing the motion of the electric car of Ex. 3, Art. 108, is repeated below.

$t =$	0	1	2	3	4	5	6	7	8	9	10	seconds
$v =$	0	2.8	5.3	7.7	9.9	11.9	13.7	15.2	16.4	17.3	18.0	miles per hour.

It is required to determine, by graphical methods, the acceleration of the car 5 sec. after starting.

Solution: The v-t graph (Fig. 230) for the entire motion is constructed; the scales chosen being 1 in. of ordinate = 20 mi/hr. and 1 in. of abscissa = 5 sec., the slope scale is unit slope = $20 \div 5 = 4$ mi/hr/sec.

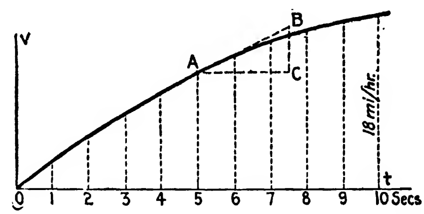


FIG. 230

A tangent line is drawn at the point on the graph corresponding to $t = 5$ sec. and a perpendicular dropped therefrom. AC and CB are measured, the slope computed, and the acceleration determined by applying the slope scale, thus:

$$AC = 0.5 \text{ in.}, \; CB = 0.24 \text{ in.}, \; CB \div AC = 0.48, \; a = 0.48 \times 4 = 1.92 \text{ mi/hr/sec.}$$

Instead of employing the slope scale as above, one might measure AC and CB directly in terms of time and velocity respectively; then the ratio $CB \div AC$ (as measured) = acceleration. Thus:

$$AC = 2.5 \text{ sec.}, \; CB = 4.8 \text{ mi/hr.}, \; a = 4.8 \div 2.5 = 1.92 \text{ mi/hr/sec.}$$

EXAMPLE 2. It is required to ascertain the distance traveled by the electric car of Ex. 3, Art. 108, during the first 10 sec. after starting.

Solution: Since in the v-t graph for the motion (Fig. 230) 1 in. of ordinate = 20 mi/hr. = 29.3 ft/sec., and 1 in. of abscissa = 5 sec., the area-scale is 1 sq. in. = 5×29.3 = 146.5 ft. The area under the curve is conveniently determined from the average ordinate.[1] The average ordinate, found by taking the mean of ordinates erected at t =

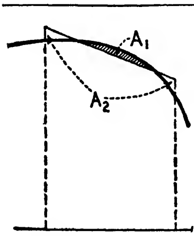


FIG. 231

[1] To determine the area under a given curve, various methods may be employed. Thus if the area to be found is divided into a large number of narrow vertical strips, each strip may be regarded as a trapezoid, and its area found by multiplying the ordinate at its middle by its width. Or the arithmetical mean of all such mid-ordinates may be taken as the average ordinate to the part of the curve in question, as in the example above. The accuracy of this method is greater the larger the number of strips taken.

Another method is to divide the area into vertical strips as before and to then replace the bounding curve by a straight line so drawn (by eye) as to make the area between this line and the curve the same above as below the curve. Thus in Fig. 231 the straight line is drawn so as to make the shaded area A_1 equal to the total unshaded area A_2. Then the area of the trapezoid is equal to the area under the curve, and the mean of the mid-ordinates of all such trapezoids represents the average ordinate to the curve.

Areas may also be found, of course, by direct measurement with a planimeter.

0.5, 1.5, 2.5, etc., is 0.545 in. The area under the curve is therefore 2×0.545 sq. in., and the distance is

$$s = 146.5 \times 1.09 = 160 \text{ ft.}$$

Instead of employing the area-scale as above, one might compute the average ordinate (for the given time interval) directly in terms of velocity; then the product of the average ordinate by the time interval = displacement. Thus the average ordinate is 16 ft/sec., the time interval is 10 sec. and the displacement is $16 \times 10 = 160$ ft.

113. Acceleration-time Graph. — The acceleration-time (a-t) graph is a curve constructed with t as abscissa and a as ordinate. Since the area under the a-t graph (between the curve, the time axis and ordinates at t_1 and t_2) is given by $\int_{t_1}^{t_2} adt$, and since $\int_{t_1}^{t_2} adt$ = increment of velocity for the interval t_1 to t_2, the area under the curve represents velocity increment according to some scale. This scale depends upon the scales used in plotting the a-t graph.

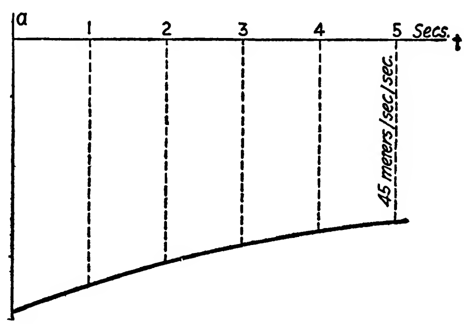


FIG. 232

EXAMPLE. Figure 232 represents a portion of the a-t graph for a projectile fired vertically upward with an initial velocity of 792 meters/sec. (the downward accelerations are plotted as negative). It is required to determine the velocity of the projectile 5 sec. after motion commenced.

Solution: Since in the a-t graph the scales are 1 in. of ordinate = 50 meters/sec/sec. and 1 in. of abscissa = 2.5 sec., the area scale is 1 sq. in. = 50×2.5 = 125 meters/sec. The area under the curve is found to be -2.19 sq. in., therefore

$$\Delta v = -2.19 \times 125 = -274 \text{ meters/sec., and}$$
$$v = 792 - 274 = 518 \text{ meters/sec.}$$

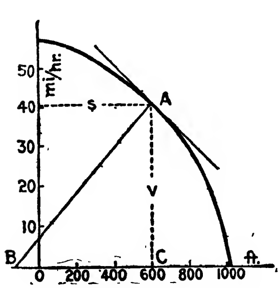


FIG. 233

114. Velocity-space Graph. — The velocity-space (v-s) graph is a curve constructed with s (position abscissa) as abscissa and v as ordinate. Neither the slope of nor the area under the curve represents any specific quantity, but the subnormal at any point of the graph represents the acceleration at the corresponding instant. For (referring to Fig. 233, the v-s graph for an air brake test on a passenger train) any subnormal as BC is given by

$$BC = AC \tan BAC = vdv/ds$$

and from Art. 109,

$$a = dv/dt = (dv/ds)\ (ds/dt) = v\ dv/ds;$$

hence BC represents a to some scale. The scale depends upon the scales employed in plotting the v-s graph.

EXAMPLE. Figure 233 represents the v-s graph for an air brake test on a passenger train; abscissas represent the distance moved by the train after the brakes were applied. It is required to determine the acceleration of the train when it had gone 600 ft. beyond the place where braking began.

Solution: A vertical is erected at C, and at A, where this vertical intersects the curve, a tangent line is drawn. AB is then drawn perpendicular to this tangent line and the subnormal BC is scaled. It is found that $BC = 0.7$ in. Now for the v-s curve one-inch ordinate $= 50$ mi/hr. and one-inch abscissa $= 1000$ ft. $= 0.19$ mi.; hence the subnormal scale is one inch $= 50^2 \div 0.19 = 13{,}150$ mi/hr/hr $= 3.65$ mi/hr/sec. Therefore the acceleration of the train at the instant in question is

$$a = 3.65 \times 0.72 = 2.63 \text{ mi/hr/sec.}$$

115. Acceleration-space Graph. — The acceleration-space (a-s) graph is a curve constructed with s as abscissa and a as ordinate. The slope of the curve does not represent any specific physical quantity, but the area under the curve (between the curve, the s-axis and ordinates at s_1 and s_2) represents half the change in the velocity-square corresponding to the change $a_2 - a_1$ or $s_2 - s_1$. For, this area is given by

$$\int_{s_1}^{s_2} a\,ds = \int_{v_1}^{v_2} v\,dv = \tfrac{1}{2}(v_2^2 - v_1^2).$$

116. Reciprocal Acceleration-velocity Graph. — The reciprocal acceleration-velocity $\left(\frac{1}{a}\text{-}v\right)$ graph for a rectilinear motion is a curve drawn upon rectangular axes so that the coördinates of any point on the curve represent corresponding, or simultaneous, values of $1/a$ and v. The area under the curve (between the curve, the v-axis and ordinates $1/a_1$ and $1/a_2$) represents the time required for the acceleration to change from a_1 to a_2, or velocity from v_1 to v_2. For, the area is given by

$$\int_{v_1}^{v_2} \frac{1}{a}\,dv = \int_{t_1}^{t_2} dt = t_2 - t_1.$$

117. Reciprocal Velocity-distance Graph. — The reciprocal velocity-distance $\left(\frac{1}{v}\text{-}s\right)$ graph for a rectilinear motion is a curve drawn upon rectangular axes so that the coördinates of any point on the curve represent corresponding, or simultaneous, values of $1/v$ and s. The area under the curve (between the curve, the s-axis and ordinates $1/v_1$ and $1/v_2$) represents the time required for the velocity to change from v_1 to v_2. For, the area is given by

$$\int_{s_1}^{s_2} \frac{1}{v}\,ds = \int_{t_1}^{t_2} dt = t_2 - t_1.$$

118. Important Special Motions. — Because they occur frequently in nature, or because they are involved in common mechanisms, certain kinds of rectilinear motion are of especial interest and importance. In the following three articles we discuss three such special motions; namely, uniformly accelerated motion, simple harmonic motion and motion of a piston.

119. Uniformly Accelerated Motion. — A point which moves with constant acceleration is said to have uniformly accelerated motion. As shown in Art. 107, the constant acceleration may be determined by dividing the velocity-increment for any time interval by the interval, that is, $a = \Delta v/\Delta t$. Since a is constant,

$$v = \int a dt = at + C$$

where C is the value of v when $t = 0$. Let v_0 denote this initial velocity; then $v = at + v_0$. Also $s = \int v dt = \int (at + v_0) dt = \frac{1}{2} at^2 + v_0 t + C$, where C is the value of s when $t = 0$. Let s_0 denote this initial position abscissa; then $s = \frac{1}{2} at^2 + v_0 t + s_0$.

It is essential to remember that *the above equations apply to uniformly accelerated motion only,* and the student is warned against the common mistake of employing them in the discussion of any other type of motion. Their application to a problem in which their use is appropriate is illustrated in Ex. 1 below.

It is easy to discuss a uniformly accelerated motion without the aid of the above derived equations, using instead elementary notions as in Ex. 2 below.

EXAMPLE 1. A stone is thrown vertically upward with an initial velocity of 100 ft/sec. A second stone is thrown upward 2 sec. later with such velocity as to just overtake the first stone at the instant the first stone starts to fall back. It is required to determine the initial velocity of the second stone.

Solution: Each stone, while in the air, is given a constant downward acceleration of 32.2 ft/sec/sec. by gravity (acceleration due to air resistance is neglected). For the first stone, then, $s_0 = 0$, $v_0 = 100$ ft/sec., and $a = -32.2$ ft/sec/sec. Therefore $v = -32.2\,t + 100$, and the stone stops ($v = 0$) when $t = 3.1$ sec. The height it reaches is $s = -(16.1 \times 3.1^2) + (100 \times 3.1) + 0 = 155$ ft. The second stone must reach this height in $3.1 - 2 = 1.1$ sec., and if v_0 be used to denote its initial velocity, the following equation applies:

$$s = (-16.1 \times 1.1^2) + (v_0 \times 1.1) + 0 = 155, \text{ whence}$$
$$v_0 = 159 \text{ ft/sec.}$$

EXAMPLE 2. The velocity of a certain train can be reduced by braking from 40 to 20 mi/hr. in a distance of 1600 ft. It is required to determine the distance within which the train could be stopped from 40 mi/hr., the retardation being assumed the same for all velocities.

Solution: Since the velocity changes uniformly, the average velocity during the reduction from 40 to 20 mi/hr. equals one-half of 40 + 20 or 30 mi/hr.; and the time

required for the reduction of velocity or travel of 1600 ft. (= 0.303 mi.) is 0.303 ÷ 30 = 0.0101 hr., or 36.4 sec. The time required to stop the train from 40 mi/hr. would be twice 36.4 or 72.8 sec.; and, inasmuch as the average velocity during the stoppage would be one-half of (40 + 0) = 20 mi/hr. or 29.3 feet per second, the distance traveled in the 72.8 sec. would be 29.3 × 72.8 = 2133 ft.

120. Simple Harmonic Motion. — If a point moves uniformly along the circumference of a circle then the motion of the projection of that point on any diameter is called a simple harmonic motion. Obviously the projection (Q) moves to and fro in its path, and travels the length of the diameter twice while the point (P) in the circumference goes once around. By *amplitude* of the s.h.m. is meant one-half the length of the path of Q, equal to the radius of the circle. By *frequency* of the s.h.m. is meant the number of complete (to and fro) oscillations of the moving point Q per unit time, equal to the number of excursions of P around the circumference per unit time. By *period* of the s.h.m. is meant the time required for one complete to and fro oscillation of the moving point Q, equal to the time required for one excursion of P around the circle.

General formulas for the velocity and acceleration of a point that has s.h.m. will now be derived, the following notation being employed:

r = amplitude (radius of the circle),
n = frequency,
$\omega = 2\pi n$ (abbreviation), ω being angle in radians swept out per unit time by OP (Fig. 234),
s = position abscissa measured from middle of path,
t = time after some convenient origin as described later,
v = velocity of the s.h.m. at the time t,
a = acceleration of the s.h.m. at the time t.

Suppose the circle (Fig. 234) to be the path of P, which moves clockwise, and consider the motion of its projection Q on the horizontal diameter AOB. Let time be dated from the instant Q is at the middle of its path (O) and moving in the positive direction (toward the right). Let θ = the angle between the radius OP and the vertical; then since P is at C when $t = 0$, $\theta = 2\pi nt = \omega t$. It is plain from the figure that $s = r \sin\theta$, and so $v = ds/dt = r\cos\theta\frac{d\theta}{dt}$. But $\theta = \omega t$ and $d\theta/dt = \omega$, and so

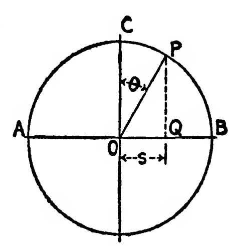


FIG. 234

$$v = r\omega\cos\theta = r\omega\cos\omega t.$$

These are the formulas for the velocity of Q in terms of θ and t respectively.

Differentiating the above expression for v with respect to t, and again placing $\theta = \omega t$ and $d\theta/dt = \omega$, it is found that

$$a = -r\omega^2\sin\theta = -r\omega^2\sin\omega t.$$

These are the formulas for the acceleration of Q in terms of θ and t respectively.

The ways in which s, v and a vary with θ (and with t as well, since θ and t are proportional) are shown by the graphs of Fig. 235 plotted with θ, (from 0° to 360°) as abscissa and s, v and a (each to some convenient scale) as ordinates. It is apparent from the v-θ graph that v is a maximum when $\theta = 0$, 180° or 360° (that is, when point Q is at the middle of its path). It is apparent from the a-θ graph that a is a maximum when $\theta = 90°$ or 270° (when the velocity is zero), and that a is zero when $\theta = 0$, 180° or 360° (when the velocity is a maximum). Also, when Q is to the right of 0 (θ between 0 and 180°) a is negative (toward the left), and when Q is to the left of 0 (θ between 180° and 360°) a is positive (toward the right). *Therefore the acceleration of a point in simple harmonic motion is always directed toward the middle of its path.*

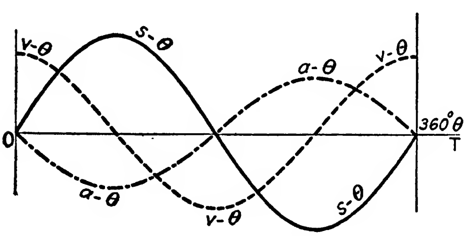


FIG. 235

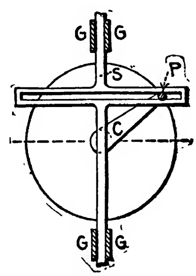


FIG. 236

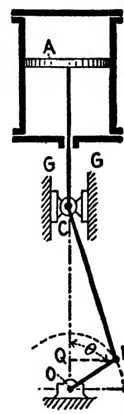


FIG. 237

Formulas for velocity and acceleration in terms of displacement do not depend on the way in which time is reckoned. Substitution in the foregoing formulas of $\sqrt{r^2 - s^2}/r$ for $\cos\theta$ and of s/r for $\sin\theta$ gives

$$v = \omega\sqrt{r^2 - s^2} = \omega r\sqrt{1 - (s/r)^2},$$

$$\text{and} \quad a = -\omega^2 s.$$

A Mechanism for Producing a Simple Harmonic Motion is represented in Fig. 236. It consists of a crank C and a slotted slider S. The slider is constrained by fixed guides G so that it can be moved to and fro only (vertically in this figure). The crank-pin P projects through the slot of the slider; hence if the crank be turned, the crank-pin presses against and moves the slider. If the crank be turned uniformly then every point of the slider executes a vertical simple harmonic motion.

121. Motion of a Piston. — Figure 237 represents a crank and connecting rod mechanism such as is used in an ordinary steam engine. OP is the crank, mounted on its shaft at O, PC is the connecting rod, C is the crosshead. (A is a piston and AC the piston rod.) When the crank is rotated the crosshead is constrained to move in a straight line by the

guides G. General formulas for the velocity of the crosshead and piston, when the crank rotates uniformly, will now be derived. Let r = the length of crank, l = length of connecting rod, $c = r/l$, n = number of revolutions of the crank per unit time (assumed constant), ω = angle in radians described by crank per unit time ($\omega = 2\pi n$), s = varying distance of the crosshead from its highest position, θ = the "crank angle" QOP, and t = time required for the crank to describe the angle θ ($= \omega t = 2\pi n t$). Obviously, there is a definite relation between s and θ (or t), and this relation is needed in order that ds/dt or v may be found. When the crosshead is in its highest position, its distance from O equals $l + r$; hence for any position,

$$s = (l + r) - CQ \mp OQ,$$

$\mp OQ$ according as the crank OP is above or below OX. Now

$$CQ = (l^2 - r^2 \sin^2 \theta)^{\frac{1}{2}} = l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}},$$

and $OQ = \pm r \cos \theta$; hence

$$s = (l + r) - l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} - r \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating the expression for s with respect to t, and noting that $d\theta/dt = \omega$, it is found that

$$v = r\omega\left(\sin \theta + \frac{c \sin 2\,\theta}{2\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}}}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the general formula for the velocity of the crosshead and piston in terms of the crank angle θ.

Differentiating the above expression for v with respect to t and noting that ω is constant and that as before $d\theta/dt = \omega$, it is found that

$$a = r\omega^2\left(\cos \theta + \frac{c \cos 2\,\theta + c^3 \sin^4 \theta}{(1 - c^2 \sin^2 \theta)^{\frac{3}{2}}}\right). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the general formula for the acceleration of the crosshead and piston in terms of the crank angle θ.

Approximate formulas for v and a can be derived as follows. As shown above

$$s = (l + r) - l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} - r \cos \theta.$$

Now $(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} = 1 - \frac{1}{2} c^2 \sin^2 \theta - \frac{1}{8} c^4 \sin^4 \theta -$ etc. (binomial expansion). And since c is generally less than $\frac{1}{3}$, the third and succeeding terms in the series are very small and negligible; hence, approximately,

$$l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} = l\,(1 - \tfrac{1}{2} c^2 \sin^2 \theta) = l\,(1 - \tfrac{1}{4} c^2 + \tfrac{1}{4} c^2 \cos 2\,\theta),$$

and

$$s = r\,(1 - \cos \theta) + \tfrac{1}{4}\, cr\,(1 - \cos 2\,\theta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiating with respect to t gives

$$v = r\omega\,(\sin \theta + \tfrac{1}{2}\, c \sin 2\,\theta). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating again gives

$$a = r\omega^2 (\cos\theta + c\cos 2\theta). \quad \text{(6)}$$

In order to furnish a comparison between the approximate formula (6) and the exact one (3), there are given in the adjoining table the values of a for the case $c = \frac{1}{3\frac{1}{2}}$ for a few values of the crank angle θ (Fig. 237).

θ	a, exact	a, approx.
0°	$+1.286\ r\omega^2$	$+1.286\ r\omega^2$
30	$+1.015$	$+1.009$
60	$+0.357$	$+0.357$
90	-0.298	-0.286
120	-0.643	-0.643
150	-0.717	-0.723
180	-0.714	-0.714

The motion of the crosshead and piston resembles very closely a simple harmonic motion when the crank is rotating uniformly. This may be seen from a comparison of the motions of C and Q, Fig. 238. In the figure, nine corresponding positions of Q and C are indicated. Thus points 0 to 8 are the positions of Q when the crank angles are 0°, $22\frac{1}{2}$°, 45°, etc., and points 0, I, II, III, etc., are the corresponding positions of C. In the lower part of Fig. 238 the paths of Q and C (with the points 1, 2, 3 and I, II, III, marked upon them) have been brought together for comparison. It is seen that the actual distances described by Q and C in any interval of time are nearly the same, and so the motion of C is nearly the same as that of Q.

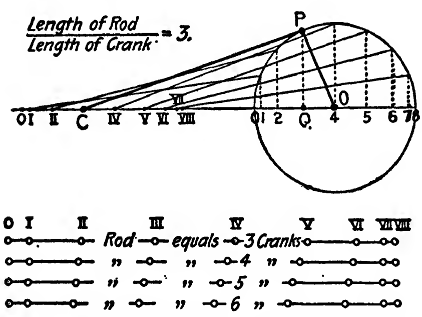


FIG. 238

The three intermediate lines in the figure are paths of C with points corresponding to 1, 2, 3, etc., for three other lengths of connecting rod. And it is apparent that the longer the rod the more nearly is the motion of the crosshead simply harmonic.

To facilitate a more complete comparison of the motions of C and Q, formulas will be derived for the position, velocity, and acceleration of Q corresponding to equations (1), (2) and (3). The variable distance of Q from P_0 (Fig. 237) will be denoted by z, then

$$z = r(1 - \cos\theta). \quad \text{(7)}$$

Differentiating with respect to t gives for the velocity of Q

$$v = r\omega \sin\theta, \quad \text{(8)}$$

and differentiating again gives for the acceleration of Q

$$a = r\omega^2 \cos\theta. \qquad (9)$$

Now compare (1) and (7), (2) and (8), and (3) and (9) and note that the formulas for the motion of C contain an "extra" term. Each of these terms depends on $c(= r/l)$, or on the "obliquity" of the connecting rod (maximum inclination of the rod to the line of stroke OC). The smaller c (the longer the rod in comparison with the crank), the smaller are the extra terms, and so the longer the rod the more nearly is the motion of the crosshead a simple harmonic one.

Figure 239 presents a comparison of the motion of the crosshead C and the motion of Q. The solid lines refer to the first motion and the dashed lines to the second; v_c is the velocity-distance (v-s) graph and a_c is the acceleration-distance (a-s graph) for the motion of C; v_q is the velocity-distance graph and a_q is the acceleration-distance graph for the motion of Q. The graphs for C were drawn for a connecting rod three cranks long ($c = 1 \div 3$). For longer rods the graphs for C would come much nearer the graphs for Q.

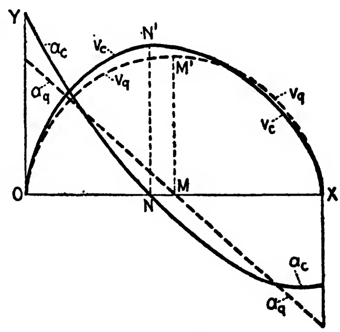


FIG. 239

§ 2. Curvilinear Motion

122. Preliminary. — Curvilinear motion was defined in Art. 102 as any motion of a point in which the path is a curved line. Certain of the definitions and relations stated for rectilinear motion apply equally to curvilinear motion, but it is necessary to extend somewhat the discussion of the various questions involved for this more general case. In rectilinear motion there are, for displacement, velocity and acceleration, but two possible directions — to right or left along the straight line path — and these two directions can be distinguished by sign. Therefore, in rectilinear motion, displacement, velocity and acceleration can be regarded as scalar quantities, varying only in magnitude and sign. In curvilinear motion displacement, velocity and acceleration may have any direction as will be explained, and so are of necessity treated as vector quantities; in the articles that follow they will be redefined and discussed on this basis.

123. Path; Position; Displacement. — A path which is a curved line may be defined by its equation referred to some system of coördinates, by description (in case it is a circle or other regular curve), or by graphical representation. The position of a point on such a given path may be specified by its coördinates, or by its distance from some fixed origin in the path, this distance being measured along the path, and, for distinction,

being considered plus when the point is on one side of the origin and minus when on the other. This distance, as in rectilinear motion, we call the position abscissa and designate by s.

The *displacement* of a point for any given interval of time is the vector joining the positions of the point at the beginning and end of the interval, the sense of the vector being from the initial to the final position.

In Fig. 240 let the curved line be the path along which a point moves; O is an assumed origin in the path. At time t_1 the point is at A; its position abscissa is then s_1. At a later time t_2 the point is at B; its position abscissa is then s_2. The displacement for the interval t_1 to t_2 is represented by the vector AB. It is seen that the displacement depends only upon the initial and final positions of the point, and not at all upon the path followed by the point in moving from one position to the other. It is apparent that the displacement is not equal to the increment Δs in the position abscissa, as it was in the case of rectilinear motion. It is also apparent that Δs is equal to the actual distance traveled by the point if the motion is not reversed during the interval, and is less than the distance actually traveled if the motion is reversed.

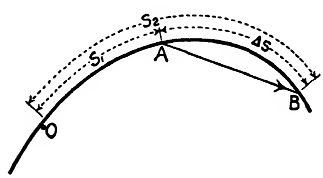


FIG. 240

124. Velocity. — By velocity of a moving point is meant the time rate at which its displacement occurs; it is a vector quantity whose direction is that of the motion. We denote velocity by V; the *magnitude* of the velocity by v. The *average velocity* for an interval of time is the displacement for that interval divided by the interval; it is a vector quantity directed like the displacement. The *velocity at a particular instant* is the limit of the average velocity for an interval that includes the instant in question as this interval is taken smaller and smaller. Since average velocity is a vector quantity, this limit is also a vector quantity and so completely represents the velocity both in magnitude and in direction. We proceed to show how to arrive at the magnitude and the direction of this vector quantity.

Let the curved line in Fig. 241 represent the path of a moving point, and let A and B be the positions of the point at the beginning and end of an interval Δt. Then the displacement is represented by vector AB; the magnitude of the average velocity is (chord AB) $\div$ Δt, and the direction of it is direction AB. The *magnitude* of the velocity of the moving point when at B is the limit of (chord AB) $\div$ Δt as A approaches B; but

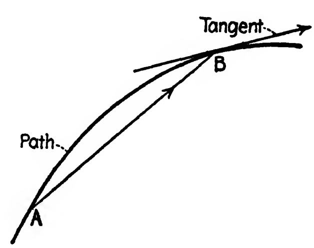


FIG. 241

$$\lim \frac{\text{chord } AB}{\Delta t} = \lim \frac{\text{arc } AB}{\Delta t} = \lim \frac{|\Delta s|}{\Delta t} = \left|\frac{ds}{dt}\right|,^1$$

which is the formula for magnitude. The *direction* of the velocity is the limiting direction of the displacement AB as A approaches B; but it is obvious from the figure that this limiting direction is the tangent to the path at B with sense indicated by the arrow.

It will be convenient to drop the signs || in the final expression above and write

$$v = \frac{ds}{dt}.$$

Then this formula applied to a numerical case gives not only the magnitude but also the sense along the tangent to the path by the sign of the computed ds/dt (see example below). The sense, however, is always apparent without reference to the sign just mentioned. The above equation can be used for finding the magnitude of the velocity in any motion in which the relation between s and t can be expressed by an equation. Differentiation gives the formula for the magnitude of V; the direction of V is given by the tangent to the path. This method of determining velocity is illustrated in the example below.

EXAMPLE. A point P starts at Q (Fig. 242) and moves in the circle shown according to the law $s = 24 - 2t^2$, s being the position abscissa in feet measured from Q (positive direction counterclockwise) and t being time in seconds after starting. It is required to determine the velocity of the point when $t = 3$.

Solution: The magnitude of the velocity is found by applying the foregoing formula for v. Since $s = 24 - 2t^2$, $v = ds/dt = -4t$. Hence when $t = 3$, $v = -12$ ft/sec. The negative sign means that the sense of the velocity along the tangent to the path is negative, that is, clockwise.

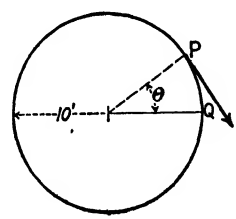


FIG. 242

The direction of the velocity is found by determining the position of the point in the path and ascertaining the direction of the tangent there. When $t = 3$, $s = 24 - 2 \times 3^2 = +6$ ft., therefore the point is 6 ft. from Q (measured counterclockwise) and the velocity is directed along the tangent to the path at P as shown. The angle V makes with the vertical is

$$\theta = 6/10 \text{ radians} = 34° \ 20'.$$

125. Speed. — As in rectilinear motion, we use the word *speed* to denote *magnitude* of velocity without regard to direction, — to express, simply, *how fast* a point is moving. Speed is a scalar quantity (without sign). An automobile running steadily at 60 miles an hour on a circular track has constant speed, but its velocity changes continuously (in direction).

The *average speed* for a given interval of time is the distance actually

[1] The symbol || means that the sign of the included quantity, if negative, is to be disregarded; thus $|\cos 135°| = 0.707$, whereas $\cos 135° = -0.707$.

traveled during that interval divided by the interval. It should be noted that in curvilinear motion the average speed for any interval is always greater than the average velocity for that interval, since obviously the distance traveled is always greater than the displacement.

126. Change of Velocity; Hodograph. — When a point has curvilinear motion its velocity changes; it necessarily changes in direction, and may also change in magnitude. The total change during any time interval is the increment in velocity, or *velocity-increment* for that interval, and is such a velocity as, added to the initial, will give the final velocity. Thus if V_1 = initial velocity, V_2 = final velocity, and ΔV = velocity-increment, then (vectorial addition and subtraction being understood)

$$V_1 + \Delta V = V_2, \quad \text{or} \quad \Delta V = V_2 - V_1.$$

The relations expressed by these equations are shown graphically in Fig. 243. The increment ΔV should be distinguished from the increment in the mere *magnitude* of the velocity, which for Fig. 243 is the difference between the mere *lengths* of the lines marked V_1 and V_2.

During the interval in question the velocity changes from V_1 to V_2, and the length of ΔV represents the amount, and the direction of ΔV the direction, of this change. Now this change does not occur all at once, the velocity changes continuously throughout the interval, and the way in which this change occurs in any given motion can best be shown by means of a certain curve now to be described.

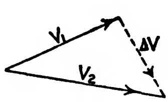


FIG. 243

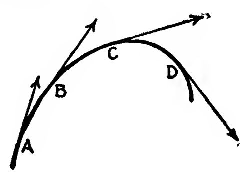


FIG. 244

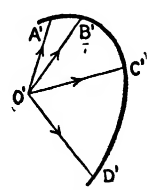


FIG. 245

Suppose a point P to move along a path represented by the curved line in Fig. 244, occupying in succession the positions A, B, C, etc. At any position, the velocity is tangent to the path there, and so the velocities are represented by tangent vectors as shown. Suppose these vectors now to be drawn from a common origin O', appearing as $O'A'$, $O'B'$, etc., Fig. 245. The free ends (A', B') of all such vectors which *could be* drawn determine a curve, as shown in the figure. This curve is called the *hodograph* for the motion of P. The points A', B', etc. (in the hodograph) are said to "correspond to" A, B, etc. (in the path). Hence, as P moves (in the path) its corresponding point P' moves in the hodograph. We denote the position abscissa of the (moving) corresponding point P' by s', and it should be understood that s' must be measured according to the scale used for representing the velocity vectors.

By means of the hodograph it is easy to determine the velocity incre-

ment for the interval during which the point undergoes any given displacement. Thus, for the displacement A to B the velocity increment is the vector (not shown) $A'B'$; for the displacement B to D, the velocity increment is vector $B'D'$, etc. And *at any instant* the way in which the velocity is changing is shown by the way in which P' is moving along the hodograph. If P' moves slowly, the velocity is changing slowly; if P' moves rapidly, the velocity is changing rapidly, and the direction in which P' moves is the direction in which the velocity change is occurring.

127. Acceleration. — By acceleration of a moving point is meant the time rate at which its velocity changes; it is a vector quantity whose direction is that in which the velocity change occurs. We denote acceleration by A; the *magnitude* of acceleration by a. The *average acceleration* for any interval of time is the increment in velocity for that interval divided by the interval; it is a vector quantity directed like the increment in velocity. The *acceleration at a particular instant* is the limit of the average acceleration for an interval that includes the instant in question as this interval is taken smaller and smaller. Since average acceleration is a vector quantity, this limit is also a vector quantity and so completely represents the acceleration both in magnitude and direction. We proceed to show how to arrive at the magnitude and the direction of this vector quantity.

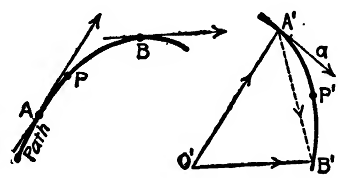


FIG. 246

Let the curve marked "Path" in Fig. 246 represent the path of a moving point P and let A and B respectively be the positions of P at the beginning and end of an interval Δt. Also suppose that the motion is such that the curve $A'B'$ is the hodograph. $O'A'$ and $O'B'$ respectively represent the velocities of P when at A and B, and the vector $A'B'$ represents the velocity increment for the interval Δt. The *magnitude* of the acceleration of the moving point when at A is the limit of (chord $A'B'$) $\div$ Δt as B is taken closer and closer to A; but

$$\lim = \frac{\text{chord } A'B'}{\Delta t} = \lim \frac{\text{arc } A'B'}{\Delta t} = \lim \frac{|\Delta s'|}{\Delta t} = \left|\frac{ds'}{dt}\right|$$

which is the formula for the magnitude. The *direction of the acceleration* is the limiting direction of the velocity increment $A'B'$ as B approaches A; but from the figure it is obvious that this limiting direction is tangent to the hodograph at A' with sense indicated by the arrow.[1]

[1] As P moves along the path the corresponding point P' (Art. 126) moves along the hodograph, and so P' may be said to have a certain velocity, the magnitude of which is measured in distance-along-the-hodograph per unit time, that is, in feet per second per second. Now ds'/dt is the magnitude, and the direction of the tangent to the hodograph the direction, of this velocity, and so the acceleration of the point P is at any instant equal (in magnitude and direction) to the velocity with which the corresponding point P' moves along the hodograph.

It will be convenient to drop the signs || in the final expression above, and write

$$a = \frac{ds'}{dt};$$

then this formula applied to any numerical case gives not only the magnitude of the acceleration ds'/dt but also the sense along the tangent to the hodograph by the sign of ds'/dt. The above formula is really not practical; a simpler method consists in determining a from its components (Art. 130 and 131). But the methods of this article are instructive, and hence we use them in the following examples. The simpler method is illustrated on the same data in Ex. 1, Art. 130, and in Ex. 1, Art. 131.

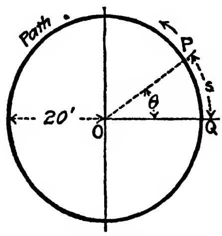


FIG. 247

EXAMPLE 1. A point P starts at Q (Fig. 247) and moves counterclockwise in the circle shown according to the law $s = 2t^3$, s being the position abscissa in feet, measured from Q, and t being the time in seconds after starting. It is required to determine the acceleration when $t = 2.4$ by graphical methods.

Solution: The velocity at each of a series of arbitrarily selected instants is determined; the computations are tabulated in the schedule below. The angle θ is equal to the angle the velocity makes with the vertical; v is the magnitude of the velocity.

t (sec.)	s (ft.)	θ (deg.)	v (ft/sec.)
1.6	8.192	23.5	15.36
1.8	11.664	33.4	19.44
2.0	16.000	45.8	24.00
2.2	21.296	61.0	29.04
2.4	27.648	79.2	34.56
2.6	35.152	100.7	40.56

From the data thus obtained the hodograph is constructed (Fig. 248). Vectors $O'A'$, $O'B'$, $O'C'$, etc., represent the velocities of P when t = 1.6, 1.8, 2.0, etc., as marked. Vectors $A'E'$, $B'E'$, $C'E'$ and $D'E'$ represent the velocity-increments for the intervals 1.6 to 2.4, 1.8 to 2.4, 2.0 to 2.4, and 2.2 to 2.4 sec. respectively. The magnitudes of these increments were scaled from the original hodograph drawing (the scale of which was one inch = 5 ft. per sec.) and are recorded in the adjoining schedule under ΔV.

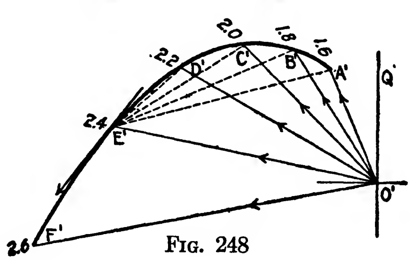


FIG. 248

Δt (sec.)	ΔV (ft/sec.)	$\Delta V/\Delta t$ (ft/sec/sec.)	Δv (ft/sec.)	$\Delta v/\Delta t$ (ft/sec/sec.)
1.6 to 2.4 = 0.8	28.75	35.9	19.20	24.0
1.8 to 2.4 = 0.6	25.15	41.9	15.12	25.2
2.0 to 2.4 = 0.4	19.55	48.9	10.56	26.4
2.2 to 2.4 = 0.2	11.45	57.2	5.52	27.6

Now the magnitude of the average acceleration for the interval 1.6 to 2.4 sec. is 28.75 ÷ 0.8 = 35.9 ft/sec/sec., and the direction of that average acceleration is $A'E'$. The magnitudes of the average accelerations for the intervals 1.8 to 2.4, 2.0 to 2.4 and 2.2 to 2.4 also are given under $\Delta V/\Delta t$; the directions of those average accelerations are respectively $B'E'$, $C'E'$ and $D'E'$.

Now the acceleration when $t = 2.4$ sec. is the limit of these average accelerations, as Δt is taken smaller and smaller but always terminating at $t = 2.4$. The magnitude of this limit, which is the magnitude of the acceleration sought, is the limit of the magnitudes of the average accelerations. This limit can be found approximately by plotting as in Fig. 249, where ordinates equal to the computed average accelerations were erected at proper points on the time-base, thus determining the solid curve *abcd*. Any other ordinate represents an average acceleration for an interval terminating at $t = 2.4$ sec., thus the ordinate at 2.1 represents the average acceleration for the interval 2.1 to 2.4. The curve *abcd* may be extended a short distance without error, and therefore the ordinate over $t = 2.4$ is approximately the limit of the values 35.9, 41.9, etc., and it represents closely the magnitude of the acceleration at $t = 2.4$ sec. This ordinate scales 66.5 ft/sec/sec. on the original drawing already mentioned. The direction of this acceleration is the limit of the directions of the average acceleration, and obviously this limit is the tangent to the hodograph at E'. On the original drawing the angle between this tangent and the horizontal is 53 degrees.

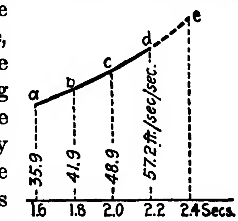


FIG. 249

For emphasis by contrast the way in which the *speed* changes during the motion under consideration will be determined. Speed-increments are listed under Δv in the schedule; average rates of change of speed for the respective time-intervals are listed under $\Delta v/\Delta t$. The limiting value of these averages, as Δt is taken smaller and smaller but always terminating at $t = 2.4$, is about 28 ft/sec/sec., and this is the rate at which the speed changes (dv/dt), at $t = 2.4$ sec.

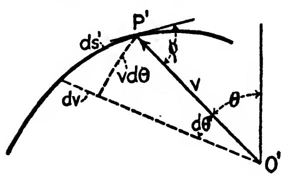


FIG. 250

EXAMPLE 2. With respect to the motion described in Ex. 1, it is required to develop general formulas for the magnitude and direction of the acceleration A of P in terms of t, and to determine this acceleration when $t = 2.4$, by analytical methods.

Solution: As in Ex. 1, let θ denote the varying angle OP makes with the horizontal; then θ is equal to the angle the velocity V makes with the vertical. Let $O'P'$ (Fig. 250) be supposed equal and parallel to V at any particular instant; then P' is a point on the hodograph, the form of which (not known and not required) will be assumed to be as shown. It is plain from the figure that

$$ds' = \sqrt{(vd\theta)^2 + (dv)^2}, \text{ and since } a = ds'/dt$$
$$a = \sqrt{v^2\,(d\theta/dt)^2 + (dv/dt)^2}.$$

Now $v = ds/dt = 6\,t^2$, whence $dv/dt = 12\,t$; also $\theta = s/20$, whence

$$d\theta/dt = \tfrac{1}{20}\,ds/dt = 0.3\,t^2.$$

Substitution of these values in the above expression for a gives

$$a = \sqrt{(36\,t^4)\,(0.09\,t^4) + 144\,t^2} = \sqrt{3.24\,t^8 + 144\,t^2},$$

which is the formula for the magnitude of A.

The formula for the direction of A is derived by noting that the angle which the

tangent to the hodograph makes with $O'P'$ (that is, the angle the acceleration makes with the velocity) is

$$\phi = \tan^{-1}\left(\frac{vd\theta}{dv}\right) = \tan^{-1}\left(v\,\frac{d\theta/dt}{dv/dt}\right) = \tan^{-1}(6\,t^2)\left(\frac{0.3\,t^2}{12\,t}\right) = \tan^{-1} 0.15\,t.^3$$

Substitution of $t = 2.4$ in the above equations for a and θ gives

$$a = 66.2 \text{ ft/sec/sec.,} \quad \text{and} \quad \phi = 64° \, 10'.$$

It is also found by substitution that $s = 27.6$ ft. and $\theta = 1.382$ radians $= 79° \, 10'$. Therefore when $t = 2.4$ the position of P is as shown in Fig. 251; its velocity is as represented by the vector V, and its acceleration is as represented by the vector A, which makes an angle with the horizontal of 53° 20′.

EXAMPLE 3. It is required to determine the acceleration of a point which describes a circle at *a constant speed.*

Solution: It is convenient to employ the relation stated in the footnote, Art. 127; namely, that the acceleration of a point P is given by the velocity of the corresponding point P' in the hodograph.

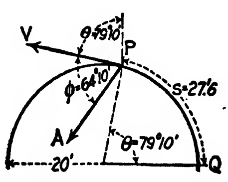


FIG. 251

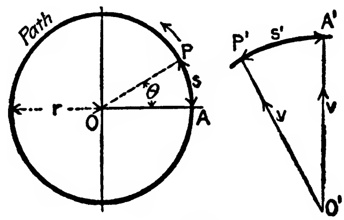


FIG. 252

Let P (Fig. 252) be the point, r = the radius of the circle, and v = the (constant) magnitude of the velocity of P. The hodograph is a circle whose radius equals v; A' corresponds to A and P' to P; and hence $A'O'P'$ equals θ. We measure the distance s (traversed by P) from A, and s' (traversed by P') from A'. Then $s'/v = s/r$, or $s' = sv/r$. Now the velocity of P' equals $ds'/dt = (ds/dt)\,(v/r) = v^2/r$, and the velocity of P' is directed along the tangent at P' (parallel to the radius OP); hence the acceleration of P is directed from P to O and its magnitude is v^2/r.

128. Axial Components of Velocity. — Since it is a vector quantity, velocity may be resolved into components. The components most useful for our purpose are those parallel to axes of coördinates x, y and z; such are called *axial components* and are denoted by V_x, V_y and V_z, their magnitudes being denoted by v_x, v_y and v_z. It will now be shown that

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt} \quad . \quad . \quad . \quad . \quad (1), (2), (3)$$

that is, *any axial component of the velocity of a point at any instant equals the rate at which the corresponding coördinate of the point is changing then.*

It is assumed for simplicity that the path of the moving point is a plane curve — in the xy plane; proof can be extended readily to include the case of a tortuous or twisted path. Let P (Fig. 253) be the moving point, v = the magnitude of the velocity of P, and α = the angle which the tangent at P makes with the x-axis. Then $v_x = v \cos \alpha$ and $v_y = v \sin \alpha$. But $v = ds/dt$, $\cos \alpha = dx/ds$, and $\sin \alpha = dy/ds$; hence

$$v_x = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt}, \quad \text{and} \quad v_y = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt}.$$

If the circumstances of a motion are such that x, y and z can be expressed by equations in t, Eqs. (1), (2) and (3) enable one to determine the axial components of the velocity by *differentiation*. The resultant velocity, if desired, can then be found by compounding these components. And if the relations between the component velocities v_x, v_y, v_z and t are given, then by *integration* the formulas for x, y and z can be determined.

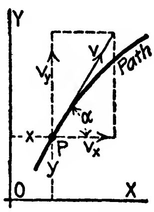


FIG. 253

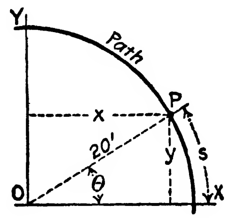


FIG. 254

EXAMPLE 1. — A point P starts at the right end of the horizontal diameter and moves counterclockwise in the circle of Fig. 254 according to the law $s = 2\,t^3$. (The motion is the same as that of Ex. 1 and 2, Art. 127.) It is required to develop the general formulas for the x and y components of the velocity, and to determine these components when $t = 2$.

Solution: It is plain from the figure that

$$x = 20 \cos\theta \quad \text{and} \quad y = 20 \sin\theta.$$

Now $\theta = s/20 = 2\,t^3/20 = 0.1\,t^3$, hence

$x = 20 \cos(0.1\,t^3)$ and $y = 20 \sin(0.1\,t^3)$, therefore

$v_x = dx/dt = -20 \sin(0.1\,t^3)\,0.3\,t^2 = -6\,t^2 \sin(0.1\,t^3)$ and

$v_y = dy/dt = 20 \cos(0.1\,t^3)\,0.3\,t^2 = +6\,t^2 \cos(0.1\,t^3)$.

The above are the general formulas for the x and y components of the velocity.

When $t = 2$,

$$v_x = -6 \times 4 \sin(0.8 \text{ radians}) = -24 \sin 45^\circ\,48' = -17.2 \text{ ft/sec.}, \quad \text{and}$$
$$v_y = +6 \times 4 \cos(0.8 \text{ radians}) = +24 \cos 45^\circ\,48' = +16.7 \text{ ft/sec.}$$

The signs indicate that the x component of the velocity is directed toward the left and that the y component is directed upward.

(The student should check the results obtained in the above solution by ascertaining, by the methods of Art. 124, the magnitude and direction of V when $t = 2$, and then resolving V into its components V_x and V_y.)

EXAMPLE 2. A point starts from the origin and moves along the curve $y = \frac{1}{4}x^2$ in such a way that the horizontal component of the velocity is always equal to 6 ft/sec. It is required to determine the velocity of the point at the end of 2 sec.

Solution: Since $v_x = 6$, $x = \int v_x dt = 6\,t + C$. Since the point starts from the origin, $x = 0$ when $t = 0$, hence $C = 0$. Therefore $x = 6\,t$. And since $y = \frac{1}{4}x^2 = 9\,t^2$, $v_y = dy/dt = 18\,t$. When $t = 2$, $v_x = 6$ and is directed toward the right, and $v_y = 36$ and is directed upward. The velocity V is therefore directed toward the right and upward. Its magnitude is

$$v = (6^2 + 36^2)^{\frac{1}{2}} = 36.5 \text{ ft/sec.}$$

The angle that V makes with the horizontal is

$$\theta = \tan^{-1} 36/6 = 80° 30'.$$

(The student should check the value obtained for θ in the above solution by determining the slope of the path, dy/dx, at the place where the point is when $t = 2$.)

129. Normal and Tangential Components of Velocity. — Since the velocity of a moving point is directed along the tangent to the path, the *tangential component* of the velocity equals the velocity itself, and the *normal component* — i.e., component perpendicular to the path — is zero.

130. Axial Components of Acceleration. — Since it is a vector quantity, acceleration can be resolved into components. When these components are taken parallel to axes of coördinates, they are called *axial components* and denoted by A_x, A_y and A_z, their magnitudes being denoted by a_x, a_y and a_z. It will be shown that

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}, \quad . \; . \; . \; (1), (2), (3)$$

that is, *any axial component of the acceleration of a point at any instant is the time rate at which the corresponding axial component of its velocity is changing then.*

We assume for simplicity that the path of the moving point is a plane curve — in the xy plane. The proof can be extended readily to include the case of a tortuous or twisted path. Let P (Fig. 255) be the moving

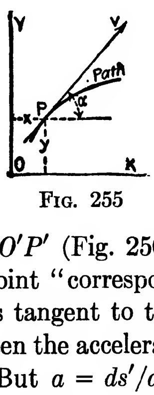


FIG. 255

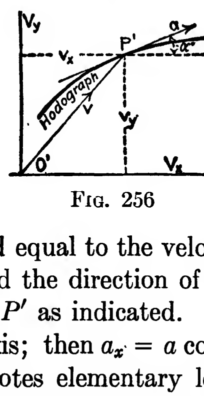


FIG. 256

point, and $O'P'$ (Fig. 256) be parallel and equal to the velocity V; then P' is the point "corresponding" to P, and the direction of the acceleration of P is tangent to the hodograph at P' as indicated. Let α' = the angle between the acceleration and the x-axis; then $a_x = a \cos \alpha'$ and $a_y = a \sin \alpha'$. But $a = ds'/dt$, where ds' denotes elementary length on the hodograph (see Art. 127); and since the coördinates of P' are v_x and v_y, $\cos \alpha' = dv_x/ds'$, and $\sin \alpha' = dv_y/ds'$. Hence

$$a_x = \frac{ds'}{dt}\frac{dv_x}{ds'} = \frac{dv_x}{dt}, \quad \text{and} \quad a_y = \frac{ds'}{dt}\frac{dv_y}{ds'} = \frac{dv_y}{dt}.$$

Substitution in (1), (2) and (3) above of the values of v_x, v_y and v_z from equations (1), (2) and (3), Art. 128, gives $a_x = d^2x/dt^2$, $a_y = d^2y/dt^2$ and $a_z = d^2z/dt^2$.

If the circumstances of the motion are such that x, y and z, or v_x, v_y and v_z can be expressed in terms of t, the above relations enable one to deter-

mine the axial components of the acceleration by *differentiation*. The (resultant) acceleration, if desired, can then be found by compounding these components. And if the relations between the component accelerations a_x, a_y, a_z and t are given, then by *integration* the formulas for v_x, v_y and v_z can be determined.

EXAMPLE 1. For the motion described in Ex. 1, Art 127, it is required to develop the general formulas for the x and y components of the acceleration, and to determine from these components the acceleration when $t = 2.4$.

Solution: It was shown in Ex. 1, Art. 128, that

$$v_x = -6\,t^2 \sin (0.1\,t^3) \quad \text{and} \quad v_y = 6\,t^2 \cos (0.1\,t^3).$$

Therefore

$$a_x = dv_x/dt = -12\,t \sin (0.1\,t^3) - 1.8\,t^4 \cos (0.1\,t^3) \quad \text{and}$$
$$a_y = dv_y/dt = 12\,t \cos (0.1\,t^3) - 1.8\,t^4 \sin (0.1\,t^3).$$

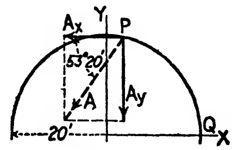


FIG. 257

The above are the general formulas for the x and y components of the acceleration.

When $t = 2.4$, substitution gives

$$a_x = -39.5 \text{ ft/sec/sec.} \quad \text{and} \quad a_y = -53.2 \text{ ft/sec/sec.}$$

The negative signs mean that A_x is directed toward the left and that A_y is directed downward; A is therefore directed toward the left and downward. Its magnitude is

$$a = (39.5^2 + 53.2^2)^{\frac{1}{2}} = 66.2 \text{ ft/sec/sec.}$$

The angle it makes with the horizontal is $\tan^{-1} 53.2/39.5 = 53°\,20'$.

For the sake of clearness the results of the solution are indicated in Fig. 257.

(The student should compare the method used in the above solution with the methods employed in Ex. 1 and 2, Art 127.)

EXAMPLE 2. A point starts from rest at the origin and moves along the parabola $y^2 = 36\,x$ (x and y in feet) in such a way that the y component of the acceleration is constant and equal to 6 ft/sec/sec. It is required to determine the position, velocity, and acceleration of the point 2 sec. after starting.

Solution: With respect to the y component of the motion, the following equations apply —

$$a_y = 6, \quad v_y = \int a_y dt = 6\,t, \quad y = \int v_y dt = 3\,t^2.$$

(The initial conditions show that the constants of integration are zero.)

With respect to the x component of the motion, the following equations apply —

$$x = \tfrac{1}{36}\,y^2 = \tfrac{1}{4}\,t^4, \quad v_x = dx/dt = t^3, \quad a_x = dv_x/dt = 3\,t^2.$$

Substitution of $t = 2$ in the above equations gives

$$\begin{aligned} x &= 4 \text{ ft.}, & y &= 12 \text{ ft.} \\ v_x &= 8 \text{ ft/sec.}, & v_y &= 12 \text{ ft/sec.} \\ a_x &= 12 \text{ ft/sec/sec.} & a_y &= 6 \text{ ft/sec/sec.} \end{aligned}$$

Therefore when $t = 2$ the point is 4 ft. to the right of, and 12 ft. above the origin. The magnitude of the velocity is $v = (8^2 + 12^2)^{\frac{1}{2}} = 14.43$ ft/sec.; it is directed to the right and up; the angle it makes with the horizontal is $\tan^{-1} 12/8 = 56°\,20'$. The magnitude of the acceleration is $a = (12^2 + 6^2)^{\frac{1}{2}} = 13.43$ ft/sec/sec.; it is directed to the right and up; the angle it makes with the horizontal is $\tan^{-1} 6/12 = 26°\,30'$.

131. Tangential and Normal Components of Acceleration. — When the components of the acceleration are taken along the tangent and normal to

the path of the moving point (at the place where the point is at the instant in question) they are called respectively tangential and normal components, and are denoted by A_t and A_n, their magnitudes being denoted by a_t and a_n. It will be shown that

$$a_t = \frac{dv}{dt}; \quad a_n = \frac{v^2}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),\ (2);$$

that is, *the tangential component of the acceleration at any instant is equal to the time rate at which the magnitude of the velocity is then changing, and the normal component of the acceleration is equal to the square of the speed divided by the radius of curvature of the path at the place then occupied by the moving point.* Also, the sense of the tangential component is the sense of the velocity when the speed is increasing and opposite to this when the speed is decreasing; the sense of the normal component is always toward the center of curvature.

Let AB (Fig. 258) be the path of a moving point P, v = magnitude of the velocity of P at A, $v + \Delta v$ = magnitude of its velocity at B, and $\Delta\theta$ = the angle between the normals (and the tangents) to the path at A and B. Also let $O'A'$ and $O'B'$ be equal to and parallel to the velocities at A and B respectively; then A' and B' are on the hodograph (Fig. 259) and angle $A'O'B' = \Delta\theta$. The acceleration of P when at A is parallel to

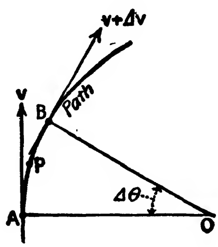


FIG. 258

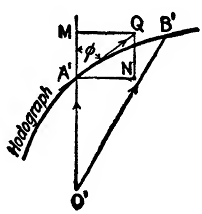


FIG. 259

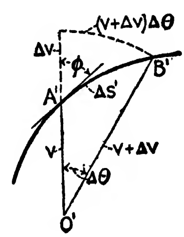


FIG. 260

the tangent $A'Q$. Let $A'Q$ represent a; then $A'M$ and $A'N$ respectively represent the tangential and normal components of a. Hence

$$a_t = a \cos\phi = \frac{ds'}{dt}\cos\phi, \quad \text{and} \quad a_n = a \sin\phi = \frac{ds'}{dt}\sin\phi.$$

Now arc $A'B' = \Delta s'$, and from Fig. 260 (which represents the same part of the hodograph as Fig. 259) it is apparent that

$$\cos\phi = \text{limit}\,\frac{\Delta v}{\Delta s'} = \frac{dv}{ds'} \quad \text{and} \quad \sin\phi = \text{limit}\,\frac{(v + \Delta v)\,\Delta\theta}{\Delta s'} = \frac{vd\theta}{ds'}$$

Also, from Fig. 258, $d\theta = ds/r$, and so

$$a_t = \frac{ds'}{dt}\frac{dv}{ds'} = \frac{dv}{dt} \quad \text{and} \quad a_n = \frac{ds'}{dt}\frac{vd\theta}{ds'} = \frac{vds}{dtr} = \frac{v}{r}\frac{ds}{dt} = \frac{v^2}{r}.$$

It will now be shown that the senses of A_t and A_n are as stated above. Let the curve MN (Fig. 261) represent the path of a moving point P (not shown) and MO a normal thereto at M. Let vector MM' represent the velocity of the point when at M, and take M as the origin for the hodograph of the motion. Now if the speed is *increasing*,[1] the velocity of P when at N, say, may be represented by MN'_1 (parallel to the tangent at N). If MN is very short, the hodograph $M'N'_1$ must lie wholly above $M'O'$ (parallel to MO). It might be convex downward, but in either case the tangent vector A_1 which represents the acceleration of P when at M must point about as shown. If the speed is *decreasing* the velocity of P when at N may be represented by MN'_2 and the hodograph $M'N'_2$ must lie wholly below $M'O'$. It might be convex downward, but in either case the tangent vector A_2, which represents the acceleration of P when at M, must point about as shown.

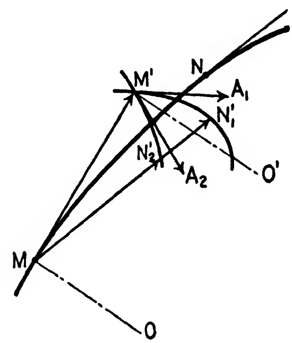


FIG. 261

It is apparent from the figure that the sense of the tangential component of A_1 is the same as the sense of the velocity, that the sense of the tangential component of A_2 is opposite to that of the velocity, and that the normal components of both A_1 and A_2 are directed toward the inside of the path.

EXAMPLE 1. For the motion described in Ex. 1, Art. 127, it is required to develop the general formulas for the tangential and normal components of the acceleration, and to determine, by means of these components, the acceleration when $t = 2.4$.

Solution: Since $s = 2\,t^3$, $v = ds/dt = 6\,t^2$; hence

$$a_t = dv/dt = 12\,t, \quad \text{and} \quad a_n = v^2/r = 1.8\,t^4.$$

The above are the general formulas for the magnitudes of A_n and A_t. The direction of A_n is obtained by noting that the angle it makes with the horizontal is $\theta = s/20 = 0.1\,t^3$; A_t is perpendicular to A_n and is directed counterclockwise because the sign of a_t is positive.

When $t = 2.4$ substitution in the above equations gives $a_t = 28.8$ ft/sec/sec. and $a_n = 59.7$ ft/sec/sec. Therefore the magnitude of the acceleration is

$$a = (28.8^2 + 59.7^2)^{\frac{1}{2}} = 66.2 \text{ ft/sec/sec.}$$

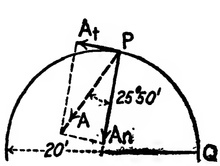


FIG. 262

The angle which A makes with its normal component (Fig. 262) is $\tan^{-1}(28.8/59.7) = 25°\ 50'$; the angle A_n makes with the horizontal is $\theta = 0.1 \times 2.4^3$ (radians) $= 79°\ 10'$; the angle A makes with the horizontal is therefore $53°\ 20'$. The results of the solution are indicated, for the sake of clearness, in Fig. 262.

(The student should compare the methods used in the above solution with those employed in Ex. 1 and 2, Art. 127, and Ex. 1, Art. 130.)

[1] Increasing continuously throughout an interval commencing some time before P is at M and terminating some time after P is at N.

EXAMPLE 2. A point P moves counterclockwise along the ellipse shown in Fig. 263 according to the law $s = \frac{1}{2} t^2$ (where s is in inches and t is in seconds), starting at such a point as to reach B when $t = 4$. It is required to determine the acceleration of P when at B.

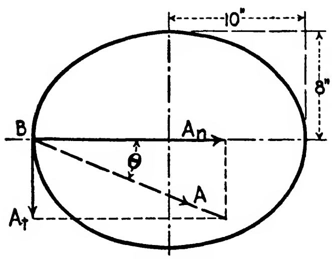


FIG. 263

Solution: Since $s = \frac{1}{2} t^2$, $v = t$, $a_t = 1$, and $a_n = v^2/r = t^2/r$. At B the radius $r = 8^2 \div 10 = 6.4$ in. Substitution of $t = 4$ and $r = 6.4$ gives $a_n = 2.5$ in/sec/sec. The component accelerations A_t and A_n are directed as represented on the figure. The magnitude of the acceleration A is

$$a = \sqrt{1^2 + 2.5^2} = 2.69 \text{ in/sec/sec}.$$

It is directed down and to the right, and the angle it makes with the horizontal is

$$\theta = \tan^{-1}(1/2.5) = 21° \, 50'.$$

CHAPTER IX

MOTION OF A RIGID BODY; RELATIVE MOTION

§ 1. Translation

132. Definition. — A translation is such a motion of a rigid body that each straight line of the body remains fixed in direction; there is no turning about of any line of the body. The coupling or side rods of a locomotive (connecting the driving wheels on either side) have a translational motion when the locomotive is running on a straight track; they would continue to have motion of translation if the track itself were made to move up or down or sideways, so long as it remained at all times parallel to its original position.

It is to be noted that the definition of motion of translation does not require that each point of the moving body have rectilinear motion, nor that the path of each point be a plane curve. It requires simply that in all successive positions the body be similarly oriented.

133. Motion of Any Point of a Body in Translation. — The motions of all points of a body in translation are alike. For, let A and B be any two points of the body, and A' and B' be the positions of those points in space at a certain instant and A'' and B'' their positions at a later instant. By definition of translation the lines $A'B'$ and $A''B''$ are parallel; and since the lines are equal in length the figure $A'B'\ B''A''$ is a parallelogram, and $A'A''$ and $B'B''$ (the displacements of A and B respectively) are equal and parallel. Since the displacements of all points of the moving body for any interval, long or short, are equal and parallel, the velocities of all points at any instant are alike, and hence also the accelerations. By displacement, velocity, and acceleration of a body having a motion of translation is meant the displacement, velocity, and acceleration respectively of any one of its points.

§ 2. Rotation

134. Definition. — A *rotation* is such a motion of a rigid body that one line of the body or of the extension of the body remains fixed. The fixed line is the *axis* of the rotation. The motion of the flywheel of a stationary engine is one of rotation and the axis of rotation is the axis of the shaft on which the wheel is mounted; the motion of an ordinary clock pendulum is one of rotation, and the axis of rotation is the horizontal line through the point of support and perpendicular to the axis of the pendulum. Obviously all points of a rotating body, except those on the axis if any, de-

scribe circles whose centers are in the axis and whose planes are perpendicular to the axis. The plane in which the center of gravity of the body moves will be called the *plane of the rotation,* and the intersection of the axis of rotation and the plane of rotation will be called *center of rotation.* All points of the body on any line parallel to the axis move alike; hence the motion of the projection of the line on the plane of the rotation represents that of all these points. And the motion of the body itself is represented by the motion of the projection of the body on the plane of rotation.

135. Angular Displacement. — By *angular displacement* of a rotating body during any time interval is meant the angle described during that interval by any line of the body perpendicular to the axis of rotation. Obviously all such lines describe equal angles in the same interval, and we select a line which cuts the axis. Let the irregular outline (Fig. 264) represent a rotating body, the plane of rotation being that of the paper, and O the center of rotation. Let P be any point and θ the angle XOP, OX being any fixed line of reference. As customarily, θ is regarded as positive or negative according as OX when turned about O toward OP moves counterclockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any rotation, then the angular displacement $= \theta_2 - \theta_1 = \Delta\theta$.

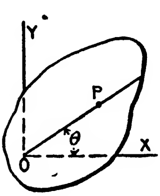


FIG. 264

136. Angular Velocity. — The angular velocity of a rotating body is the time-rate at which its angular displacement occurs; or, otherwise stated, it is the time-rate at which any line of the body perpendicular to the axis of rotation describes angle. We denote angular velocity by ω.

If the body is rotating uniformly (describing equal angles in all equal intervals of time) then the angular velocity is constant, and may be computed by dividing the angular displacement for any interval of time by the interval.

If the body is not rotating uniformly, then the angular displacement divided by the interval gives the *average angular velocity* for the interval; the *angular velocity at any particular instant* is the limit of the average angular velocity for an interval that includes the instant in question, as this interval is taken shorter and shorter. In the calculus notation this limit is $d\theta/dt$, hence

$$\omega = \frac{d\theta}{dt}.$$

It is evident from the equation that ω is positive or negative according as $d\theta$ is positive or negative, and so an angular velocity is regarded as having sign, the sign being determined by the direction of the rotation. In conformity with the rule for sign given in Art. 135, the angular velocity is considered positive when the rotation is counterclockwise and negative when the rotation is clockwise.

The above equation implies that the unit angular velocity is that of a body rotating uniformly and making a unit angular displacement in unit time. There are several such units; thus, one revolution per minute, one degree per hour, one radian per second, etc.[1] The last is the one generally used herein.

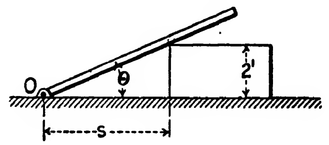


Fig. 265

Example. A straight bar is pinned at one end to a horizontal floor and rests upon a block as shown in Fig. 265. The block can be pushed to and fro along the floor, thus causing the bar to rotate about O. It is required to develop a general formula for the angular velocity of the bar and to determine this angular velocity when the block is 4 ft. from O and is moving toward the left at 6 ft/sec.

Solution: If s denote the distance in feet from O to the near face of the block, it is evident from the figure that $\theta = \tan^{-1} 2/s$. Therefore

$$\omega = \frac{d\theta}{dt} = \left(\frac{1}{1 + (2/s)^2}\right)\left(-\frac{2}{s^2}\right)\frac{ds}{dt},$$

and since $ds/dt = v$, the velocity of the block, this reduces to

$$\omega = \left(\frac{-2}{s^2 + 4}\right) v \text{ (rad/sec.)}.$$

This is the general formula for ω in terms of s and v (the position abscissa and velocity, respectively, of the block). When v is positive (block moving toward the right), ω is negative (bar rotating clockwise); when v is negative (block moving toward the left), ω is positive (bar rotating counterclockwise).

When $v = -6$ and $s = 4$, substitution gives $\omega = 0.6$ rad/sec.

137. Angular Acceleration. — The angular acceleration of a rotating body is the time-rate (of change) of its angular velocity. We denote angular acceleration by α.

If the angular velocity changes uniformly, then the angular acceleration is constant and may be computed by dividing the increment in angular velocity for any interval of time by the interval.

If the velocity does not change uniformly, then the increment divided by the interval gives the *average angular acceleration* for the interval; the *angular acceleration at any particular instant* is the limit of the average angular acceleration for an interval that includes the instant in question, as this interval is taken shorter and shorter. In the calculus notation this limit is $d\omega/dt$, hence

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

It is evident from the equation that α is positive or negative according as $d\omega$ is positive or negative, and so an angular acceleration is regarded as having sign. The angular acceleration is positive when positive angular velocity is being taken on; it is negative when negative angular velocity

[1] For dimensions of a unit angular velocity, see Appendix A.

is being taken on. And so angular acceleration has direction, or sense, — that of the angular velocity that is being taken on.

The above equation implies that the unit angular acceleration is that of a body whose angular velocity is changing uniformly and so that unit angular velocity-change occurs in unit time. One revolution per second per second, one radian per second per second, etc., are such units.[1]

EXAMPLE. It is required to develop the general formula for the angular acceleration of the bar described in the example of Art. 136, and to determine this angular acceleration when the block is 4 ft. from O and moving toward the left with a velocity of 6 ft/sec. and an acceleration toward the left of 3 ft/sec/sec.

Solution: It was found in the example of Art. 136 that $\omega = \left(\frac{-2}{s^2+4}\right) v$. Differentiating with respect to t, and noting that both s and v are variable and that $dv/dt = a$, the acceleration of the block, it is found that

$$\alpha = \left(\frac{4\,s}{(s^2+4)^2}\right) v^2 + \left(\frac{-2}{s^2+4}\right) a \text{ (rad/sec/sec.}$$

This is the general formula for the angular acceleration of the bar, and it will be noted that α depends upon all three quantities s, v and a, that is, upon the position, velocity and acceleration of the block.

When $s = 4$, $v = -6$, and $a = -3$ (conditions stated above), substitution gives $\alpha = 1.74$ rad/sec/sec.

138. Motion of Any Point of a Body in Rotation. — There are simple relations between the linear velocity v and linear acceleration a of any point P of a rotating body and the angular velocity and acceleration of the body. Let r = the distance of P from the axis of rotation, s = distance traveled by P in any time from some fixed point in the path of P, and θ = the angle described by the radius to P in that same time. Then $s = r\theta$ *if* θ *be expressed in radians;* $ds/dt = r\,d\theta/dt$, or

$$v = r\omega. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Differentiating again, we find that $dv/dt = r\,d\omega/dt$, or

$$a_t = r\alpha; \quad \text{also} \quad a_n\ (= v^2/r) = r\omega^2. \quad \ldots\ldots \quad (2), (3)$$

Here a_t and a_n mean the tangential and normal components of the acceleration of P (Art. 131).

It should be noted that consistent units must be employed in the above equations. Thus if v is in feet per second and a in feet per second per second, r should be in feet, while ω should be in *radians* per second, and α in *radians* per second per second.

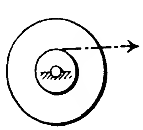


FIG. 266

EXAMPLE. Figure 266 represents a wheel (diameter 4 ft.) and attached drum (diameter 1.5 ft.) mounted upon an axle. By means of a rope wound about the drum, the wheel is made to rotate. It is required to determine how great a velocity and tangential acceleration a point on the perimeter of the wheel has when the rope is being drawn out at the rate of 30 ft/sec. and with a linear acceleration of 12 ft/sec/sec. (The thickness of the rope may be neglected.)

[1] For dimensions of a unit angular acceleration, see Appendix A.

Solution: Obviously the speed of a point on the perimeter of the drum is the same as that of a point on the rope, therefore the angular velocity of the drum is $\omega = 30 \div 0.75 = 40$ rad/sec. This is likewise the angular velocity of the wheel. The magnitude of the velocity of a point on the perimeter of the wheel is therefore $v = 2 \times 40 = 80$ ft/sec.

Obviously the tangential acceleration of a point on the perimeter of the drum is just as great as the linear acceleration of a point on the straight portion of the rope. Therefore the angular acceleration of the drum is $\alpha = 12 \div 0.75 = 16$ rad/sec/sec. This is likewise the angular acceleration of the wheel. Therefore the magnitude of the tangential acceleration of a point on the perimeter of the wheel is $a_t = 2 \times 16 = 32$ ft/sec/sec.

§ 3. Plane Motion

139. Definition. — Plane motion is a motion in which every point of the moving body remains at a constant distance from a fixed plane. Each point of the body moves in a plane; that is, its motion is uniplanar. By *plane of the motion* is meant the plane in which the center of gravity of the body moves. The wheels of a locomotive running on a straight track have plane motion; also a book which is slid about in any way on the top of a table. A translation (Art. 132) may or may not be a plane motion; a rotation about a fixed axis (Art. 134) is always a plane motion.

In a plane motion all points of the moving body which lie on a perpendicular to the plane of the motion move alike, and the motion of the projection of this line on the plane of the motion correctly represents the motion of all the points. So also the motion of the projection of the moving body upon the plane of the motion correctly represents the motion of the body itself. Thus we have a plane figure (the projection just mentioned) moving in a plane representing a plane motion of a body; and since the motion of the plane figure is uniplanar, the motion of the body is called uniplanar. Hereafter, we will sometimes refer to the projection of the body as the body itself.

140. Angular Displacement, Velocity and Acceleration. — By *angular displacement* of a body whose motion is plane is meant (as in rotation) the angle described by any line of the body which is in the plane of the motion. Obviously all such lines describe equal angles in the same interval of time. As in rotations also, displacements are regarded as positive or negative according as they are due to counterclockwise or clockwise turning of the body. Let the irregular outline (Fig. 267) represent the projection of the moving body on the plane of the motion, AB a fixed line of the projection, and OX a fixed reference line; also let θ denote the angle XOA, it being regarded as positive or negative according as OX, when turned about O toward AB, turns counterclockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any motion of the body, then the angular displacement $= \theta_2 - \theta_1 = \Delta\theta$.

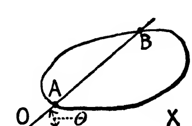


FIG. 267

If a body has a plane motion, its *angular velocity* is the time-rate at which its angular displacement occurs, and its *angular acceleration* is the time-rate at which its angular velocity changes. These definitions are precisely similar to those of the angular velocity and acceleration of a rotation about a fixed axis (Art. 136 and 137); hence the expressions, units and rules of signs given in those articles hold also for any plane motion. The expressions are

$$\omega = d\theta/dt \quad \text{and} \quad \alpha = d\omega/dt = d^2\theta/dt^2,$$

ω and α denoting angular velocity and acceleration of the moving body respectively.

141. Plane Motion Regarded as a Combination of Translation and Rotation. — *Any uniplanar displacement of a body can be accomplished by means of a translation of the body followed by a rotation, or vice versa.* Thus let $A_1B_1C_1$ (Fig. 268) be one position of a body ABC, and $A_2B_2C_2$ a sub-

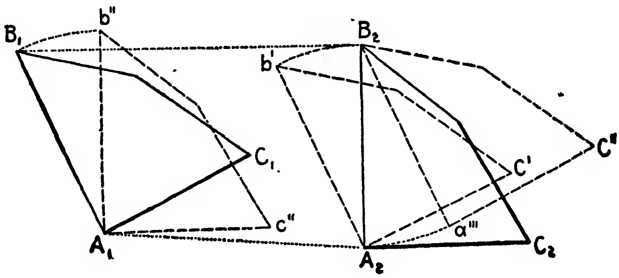


FIG. 268

sequent position. By means of a translation the body can be displaced so that one of its points is put into its final position; thus a translation to $A_2b'c'$ puts A into its final position. Then a rotation of the body about A_2 puts the body into its final position. Or, by means of a rotation we can put the body into an intermediate position $A_1b''c''$ so that each line in it will be parallel to its final position (in $A_2B_2C_2$); and then the body may be put into its final position by a translation. Obviously, the translation and rotation might be performed simultaneously. The point (or axis) of the body about which we imagine the rotation to occur is called *a base point* (or *base axis*).

Figure 268 also represents a displacement from $A_1B_1C_1$ to $A_2B_2C_2$, accomplished with B as base point. A translation puts the body into the position $B_2a'''C''$, and a suitable rotation about B_2 puts it into the final position $B_2A_2C_2$.

It is clear that the amount of the translation component depends on the base point; thus A_1A_2 is the translation for A as base point, while B_1B_2 is the translation for B as base point. But the amount of the rotation component does not depend on the base point; thus the rotation is the angle

$b'A_2B_2$ for A as base point, and it equals the angle $a'''B_2A_2$ which is the rotation for B as base point.

The successive small displacements of ABC from $A_1B_1C_1$ to $A'B'C'$, $A''B''C''$, etc., to $A_2B_2C_2$ (Fig. 269) already mentioned (and which, all together, approximate to a continuous motion of ABC in which all points of the body move along smooth curves), can each be made by a small simultaneous translation and rotation. And if we take some one point as base point for all these small displacements then we may regard the motion as a continuous combined or simultaneous translation and rotation, the translation being like the motion of the base point and the rotation being about that point.

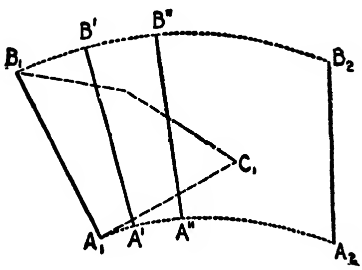


FIG. 269

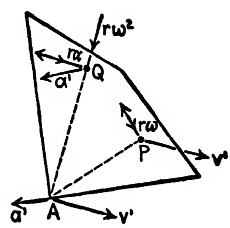


FIG. 270

142. Motion of Any Point of the Body. — In accordance with the view of plane motion set forth in the preceding article the velocity of any point of the moving body at any particular instant consists of two components, one corresponding to the translation and one to the rotation. Thus let A (Fig. 270) be the chosen base point, v' = the velocity of A for the position of the body shown, and ω = the angular velocity of the body at the instant under consideration. Then the first component of the velocity of any point P equals v' and is directed like v'; the second component equals $r\omega(r = AP)$ and is directed at right angles to AP, the sense depending on the sense of ω (clockwise or counterclockwise).

The acceleration also of any point consists of two components, one corresponding to the translation component of the motion and one to the rotation. Thus let a' be the acceleration of the base point, and α = the angular acceleration of the body. Then the first component of the acceleration of any point Q equals a' and is directed like a'; the second component we describe by means of two components, as in a rotation about a fixed axis (see Art. 138), one of which (the normal component) is directed along QA and the other (the tangential component) is at right angles to QA. The normal component equals $r\omega^2(r = AQ)$ and is always directed from Q to A, toward the base point or center of the rotational component; the tangential component equals $r\alpha$, and obviously its sense depends on the sense of the angular acceleration.

EXAMPLE. In Fig. 271 AB represents the crank and BC the connecting rod of an engine; their lengths are respectively $1\frac{1}{2}$ and 5 feet. It is required to determine the velocity and acceleration of the crosshead C when the engine is running uniformly at 100 r.p.m. (clockwise rotation) and the connecting rod and crank are in the position corresponding to $\theta = 35°$.

Solution: Point B is selected as base point, because it is the one point on the rod whose motion is known. Let V' denote the velocity of B, A_n' the normal component of

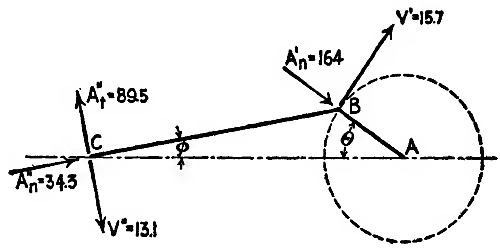


FIG. 271

its acceleration (the tangential component is zero), and ω' the angular velocity of the crank AB. Let V'' denote the velocity of C due to its rotation about B, A_t'' and A_n'' the tangential and normal components of its acceleration due to this rotation, and ω and α the angular velocity and acceleration respectively of the connecting rod BC. These components of velocity and acceleration are indicated on the figure (the *senses* of V', A_n', V'' and A_n'' are evident, the sense of A_t'' is assumed); they will now be determined.

Let θ denote the angle the crank AB makes with the horizontal and ϕ the angle the connecting rod BC makes with the horizontal. From the figure it is evident that

$$\phi = \sin^{-1}(1.5 \sin\theta/5) = \sin^{-1}(0.3 \sin\theta).$$

Therefore (noting that $d\theta/dt = \omega'$)

$$\omega = \frac{d\phi}{dt} = \left(\frac{0.3\cos\theta}{\sqrt{1-(0.3\sin\theta)^2}}\right)\omega',$$

and (noting that ω' is constant)

$$\alpha = \frac{d\omega}{dt} = \left(\frac{-0.3\sin\theta\sqrt{1-(0.3\sin\theta)^2} + 0.027\sin\theta\cos^2\theta/\sqrt{1-(0.3\sin\theta)^2}}{1-(0.3\sin\theta)^2}\right)(\omega')^2.$$

Substitution of $\theta = 35°$ in the above equations gives

$$\omega = 2.62 \text{ rad/sec.} \quad \text{and} \quad \alpha = -17.9 \text{ rad/sec/sec.}$$

(The signs show that the rod is turning counterclockwise and that its angular acceleration is clockwise; since α is clockwise, the sense of A_t'' is as assumed.)

Since the radius of C with respect to the base point B is 5 ft., $v'' = 5 \times 2.62 = 13.1$ ft/sec., $a_t'' = 5 \times 17.9 = 89.5$ ft/sec/sec., and $a_n'' = 5 \times 2.62^2 = 34.3$ ft/sec/sec. Since B is describing a 3 ft. circle at a uniform speed of 100 r.p.m., $v' = 15.7$ ft/sec. and $a_n' = 164$ ft/sec/sec. For clearness these values are recorded on the figure.

The actual velocity V of C can now be determined by compounding V' and V''. This is most easily done by summing up along the horizontal and along the vertical, thus (noting that $\phi = 9° 54'$ when $\theta = 35°$)

$v_x = 13.1 \sin 9° 54' + 15.7 \sin 35° = 11.2$ ft/sec. and

$v_y = -13.1 \cos 9° 54' + 15.7 \cos 35° = 0$ (as is evident from the circumstances of the motion).

The actual acceleration A of C is similarly determined by compounding A_n', A_t'' and A_n'', thus:

$a_x = 164 \cos 35° - 89.5 \sin 9° 54' + 34.3 \cos 9° 54' = 152$ ft/sec/sec.

$a_y = -164 \sin 35° + 89.5 \cos 9° 54' + 34.3 \sin 9° 54' = 0$ (as is evident from the circumstances of the motion).

When the connecting rod is in the position shown, then, the crosshead C has a velocity toward the right of 11.2 ft/sec. and an acceleration toward the right of 152 ft/sec/sec.

(The student should check the results obtained above by means of the formula (Eq. 2) developed in Art. 121. It will be noted that the method employed in this example is the lengthier, but it has this advantage, namely, that when the expressions for ω and α have been developed, the velocity and acceleration of *any point* on the connecting rod can readily be found. This is of importance in investigations as to the stresses set up in members of this type by the great accelerations occurring in high speed machinery.)

143. Plane Motion Regarded as a Rotation about a Moving Axis. — *Any uniplanar displacement of a body can be accomplished by means of a single rotation.* Thus consider the displacement of ABC from the position $A_1B_1C_1$ to $A_2B_2C_2$ (Fig. 272). The point A can be brought from A_1 to A_2 by means of a rotation of AB about any point on the perpendicular bisector aO (of A_1A_2); and B can be brought from B_1 to B_2 by means of a single rotation of AB about any point on the perpendicular bisector bO (of B_1B_2). If the intersection of the bisectors is taken for the center of rotation of both A and B, then the amounts of the rotations (angles A_1OA_2 and B_1OB_2) are equal; hence, the line AB (and body ABC) can be displaced from one

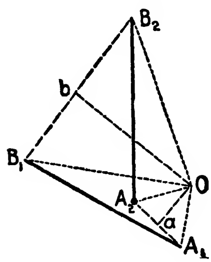


FIG. 272

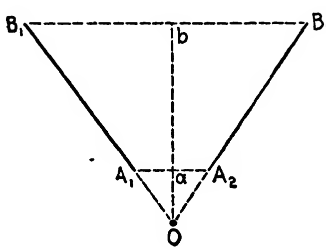


FIG. 273

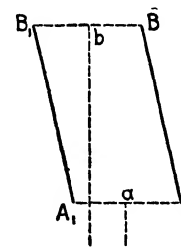


FIG. 274

position to any other (uniplanar displacement) by means of a single rotation as stated.

In case the two bisectors coincide (Fig. 273), then the angles B_1 and B_2 are equal and hence the lines A_1B_1 and A_2B_2 extended intersect on the bisector ab extended; this extension is the center of rotation C which would displace AB from A_1B_1 to A_2B_2. In case the bisectors are parallel (Fig. 274) the center of rotation is "at infinity," and the displacement is a translation; thus a uniplanar translation may be regarded as a rotation about a center at infinity.

The actual continuous motion of AB from one position A_1B_1 to another A_2B_2 (in which A and B describe smooth curves) can be closely duplicated

by a succession of rotations of AB from A_1B_1 (Fig. 269) into successive intermediate positions $A'B'$, $A''B''$, etc., until A_2B_2 is reached. Each small rotation is made about a definite center O', O'', etc. (not shown). The closer these intermediate positions are taken (and the more numerous and closer the centers of rotation O', O'', etc.) the more nearly do the successive rotations reproduce the actual continuous motion. "In the limit," the actual motion is reproduced by the rotations, the centers of rotation forming a continuous line. Thus we may regard any uniplanar motion of a body as consisting of a continuous rotation about a center which, in general, is continuously moving. The position of the center O about which the moving body is rotating at any instant is called the *instantaneous center* of the motion for the particular instant or position (of the body) under consideration, and the line through that center and perpendicular to the plane of the motion is called the *instantaneous axis* of the motion for that instant. (See next article for graphical method of locating instantaneous centers.)

In general, the instantaneous center moves about in the body and in space. Its path in the body is called the *body centrode;* its path in space the *space centrode.* Thus, in the case of a wheel rolling on a plane, the instantaneous center at any instant is the point of contact between the wheel and plane; the successive instantaneous centers on the wheel trace or mark out the circumference and this line is the body centrode; the successive instantaneous centers in space trace or mark out the track and this line is the space centrode. It can be shown that any plane motion may be regarded as a rolling of the body centrode on the space centrode.

144. Velocity of Any Point; Instantaneous Center. — In a rotation about a fixed axis the velocities of all points of the body are proportional to the distances of the points from the axis of rotation, and the velocities are respectively normal to the perpendiculars from the points to the axis (Art. 138); the velocity of any particular point is given by $v = r\omega$, where $v =$ the velocity of the point, $r =$ the distance of the point from the axis, and $\omega =$ the angular velocity of the body. So, too, in the case of any uniplanar motion, the velocities of all points of the body at any particular instant are proportional to the distances of the points from the instantaneous axis (corresponding to that instant); the velocities are respectively normal to the perpendiculars from the points to the instantaneous axis; and the velocity v of any particular point is given by $v = r\omega$, where $r =$ the distance from the point to the axis and $\omega =$ the angular velocity of the body.

By means of the foregoing velocity relations, we can locate the instantaneous center for any given position of the moving body if the directions of the velocities of two of its points are given; and then if the value of one velocity is given we can compute the angular velocity of the body and the velocity of any other point.

From the preceding paragraph it follows that if through any point of the moving body a line be drawn perpendicular to the direction of the velocity of that point, then the instantaneous center (for the position of the body under consideration) lies somewhere on that perpendicular. Hence, the instantaneous center is at the intersection of such perpendiculars drawn for any two points of the body.

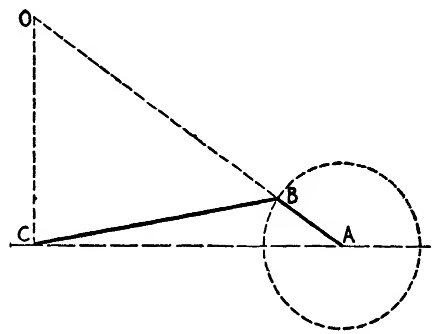


FIG. 275

EXAMPLE. It is required to determine the velocity of the crosshead C of the connecting rod mechanism described in the example of Art. 142.

Solution: Graphical methods are convenient and will be employed. Figure 275 represents a scale drawing of the mechanism in the position described ($\theta = 35°$). Since the velocity of the point B of the rod is along the tangent to the crankpin circle at B, the instantaneous center of the connecting rod is on the normal to the tangent at B, that is on AB or its extension; and since the velocity of the point C of the rod is along AC, the instantaneous center is on the normal to AC. Hence the instantaneous center is at the intersection O. From the drawing it is found that $OB = 6$ ft. and $OC = 4.27$ ft.

Now the speed of B is $100 \times 3\,\pi/60 = 15.7$ ft/sec., hence the angular velocity of the connecting rod $15.7 \div OB = 2.62$ radians/sec. The magnitude of the velocity of C is therefore

$$v = OC \times 2.62 = 11.2 \text{ ft/sec.}$$

§ 4. Spherical Motion and General Three Dimensional Motion of a Rigid Body

145. Spherical Motion Defined. — Spherical motion means motion of a rigid body with only one point of the body fixed. Each point of the body, excepting the fixed one, moves on the surface of a sphere, whence the name spherical motion.

146. Spherical Motion Regarded as a Rotation about a Moving Axis. — *Any spherical displacement of a body can be accomplished by means of a rotation about some line of the body passing through the fixed point, and fixed in space.* Proof: — Evidently, we may describe any position of the body by describing the positions of two of its points, not in line with the fixed point. Let A and B denote two such points, equally distant from the fixed point O; then during any motion of the body, A and B move on the surface of the same sphere. Let OA_1B_1 (Fig. 276) be one position of the body, and OA_2B_2 another. Then we are to prove that the points A and B could be brought from A_1B_1 to A_2B_2 by means of a single rotation about some fixed line through O. Let the lines A_1B_1 and A_2B_2 be arcs of great circles of the sphere mentioned; these arcs are equal since A and B are points of a rigid

body. The lines A_1A_2 and B_1B_2 are arcs of great circles; M and N bisect these arcs. MR and NR are great circles perpendicular to A_1A_2 and B_1B_2 respectively. In general two such great circles do not coincide but intersect at two points, R and S. The diameter ROS is the axis, rotation about which would produce the given displacement, proven presently. Let A_1R, A_2R, B_1R, and B_2R be arcs of great circles. Since A_1A_2R and B_1B_2R are isosceles triangles, $A_1R = A_2R$ and $B_1R = B_2R$; and, as already stated, $A_1B_1 = A_2B_2$. Hence the triangles RA_1B_1 and RA_2B_2 are equal, and the angle $A_1RB_1 = A_2RB_2$. Finally,

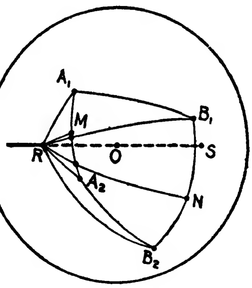


FIG. 276

$$A_1RA_2 = A_1RB_2 - A_2RB_2 = A_1RB_2 - A_1RB_1 = B_1RB_2.$$

Hence a rotation of the great circles A_1R and B_1R about RS of an amount equal to the angle A_1RA_2 would displace A from A_1 to A_2 and B from B_1 to B_2.

Imagine any actual continuous spherical motion of a body, in which the two points A and B of the body are displaced from A_1 to A_2 and B_1 to B_2 respectively. Let A', A'', etc., be several intermediate positions of A, and let B', B'', etc., be corresponding intermediate positions of B. As already shown, the displacements of AB from A_1B_1 to $A'B'$, from $A'B'$ to $A''B''$, from $A''B''$ to $A'''B'''$, etc., might be accomplished by single rotations about definite fixed lines $R'OS'$, $R''OS''$, $R'''OS'''$, etc. If a large number of intermediate positions $A'B'$, $A''B''$, etc., be assumed, and if the successive rotations be accomplished in times equal to the times required for the actual displacements in the continuous motion, then the succession of rotations would closely resemble the actual continuous motion. The more numerous the intermediate positions, and the more numerous the succession of single rotations, the more closely would the succession resemble the actual motion. "In the limit," the succession would reproduce the actual motion; hence we may regard any spherical motion of a body as consisting of a continuous rotation about a line through the fixed point, the line continually shifting about in the body and in space. The line about which the body is rotating at any instant is the *instantaneous axis* (of rotation) at or for that instant.

147. Angular Velocity in Spherical Motion. — At any particular instant of a spherical motion, the body is rotating about the instantaneous axis at a definite rate; this rate is called the angular velocity of the body at that instant. We generally denote magnitude of angular velocity by ω.

In a rotation about a fixed axis, the (linear) velocity of any point of the body equals the product of the angular velocity and the perpendicular distance (or radius) from the point to the axis; and the direction of the

linear velocity is perpendicular to the plane of the radius and the axis. So too in a spherical motion, the linear velocity of any point of the body at any instant equals the product of the angular velocity at that instant and the radius (perpendicular from the point to the instantaneous axis for that instant); the direction of that velocity is perpendicular to the plane of the radius and the axis.

Any angular velocity ω may be represented by means of a vector laid off on the corresponding instantaneous axis; the length of the vector is made equal to ω according to some convenient scale, and the sense of the vector indicates the direction of the rotation according to some convention. We use this: the sense of the vector should be the same as the advance of a right-hand screw when turned in a fixed nut in the direction of ω.

148. Composition and Resolution of Angular Velocities. — Imagine a body A (Fig. 277) to be rotating about a line l_1, fixed in a body B; and that B is rotating about a line l_2 which intersects l_1 and is fixed in a body C. For convenience we regard the actual motion of A as the resultant of the component rotations about l_1 and l_2. We show presently that (i) the resultant motion is spherical, and (ii) the angular velocity of the motion can be ascertained by the parallelogram law — that is, if OM and ON (Fig. 278) represent the angular velocities of the component rotations, then OR is the instantaneous axis and represents the angular velocity of the resultant motion of A.

(i) That the absolute motion of A is spherical seems almost self evident; for clearly the point O of A extended does not move at all, since O is also a point of C extended, and O *appears* to be the only point of A which is fixed. That O *is* the only fixed point of A can be shown from the parallelo-

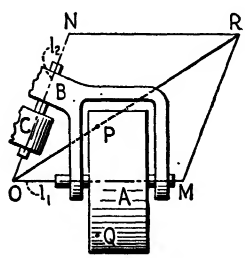


FIG. 277

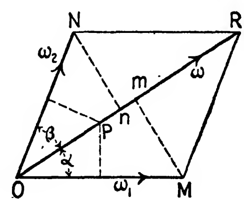


FIG. 278

gram construction (the validity of which is proved below). Consider any other point P of A which lies on the instantaneous axis at the instant under consideration, that is when that axis is OR, *in* the plane of the paper. At a later instant, l_1 and hence the instantaneous axis also are out of the plane of the paper. The point P is then either on the axis or not. If on the axis P must have moved during the interval; if not on the axis it has velocity at the later instant. Hence, in either case P is not fixed during the interval, and O is the only fixed point of A (extended).

(ii) Let ω_1 and ω_2 respectively denote the angular velocities of the component rotations, about l_1 and l_2, and ω the angular velocity of the resultant rotation. Let P (Fig. 278) be any point of A on the diagonal OR extended if necessary. On account of rotation about l_1, P comes up out of the paper and on account of rotation about l_2 it goes down; that is the velocities of P due to these rotations are colinear but opposite. The values of these component velocities respectively are

$$(OP \sin \alpha)\omega_1 = OP(OM \sin \alpha) = OP \times Mm,$$

and

$$(OP \sin \beta)\omega_2 = OP(ON \sin \beta) = OP \times Nn.$$

But Mm and Nn are equal, and hence these colinear and opposite velocities of P are equal; and so the resultant velocity of any point of A on the diagonal OR is zero.

Take any point Q of the body A and let q_1, q_2 and q denote its distances from OM, ON and OR respectively (Fig. 279). The component velocities of Q due to the rotations about l_1 and l_2 respectively are

$$q_1\omega_1 = q_1 OM \quad \text{and} \quad q_2\omega_2 = q_2 ON,$$

and the resultant velocity of Q is $q\omega$. The component velocities are in the same direction; hence the resultant velocity equals their sum or

$$q\omega = q_1 OM + q_2 ON.$$

Now

$$q_1 OM = 2\,(\text{area } OQM) = OQ \times Mm,$$

and

$$q_2 ON = 2\,(\text{area } OQN) = OQ \times Nn;$$

hence

$$q\omega = OQ\,(Mm + Nn) = OQ \times Rr = 2\,(\text{area } OQR) = qOR,$$

or

$$OR = \omega,$$

that is the diagonal OR represents the resultant angular velocity to the same scale as OM and ON represent ω_1 and ω_2 respectively.

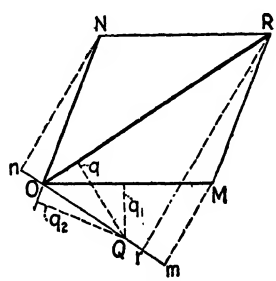


FIG. 279

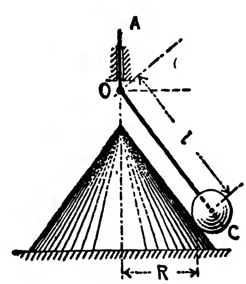


FIG. 280

EXAMPLE. The sphere (Fig. 280) rests against a cone and is suspended from the end of a vertical shaft by means of a rod extending into and rigidly fastened to the sphere. The shaft and rod are connected by a Hooke's (flexible) joint, at O. The entire apex angle of the cone is 140 deg., the diameter of the sphere is 4 ft., and R = 4 ft. The shaft is rotated so that the center of the sphere makes 30 trips per minute around the cone, counterclockwise when viewed from above. Required the angular

velocity of the sphere; also components of that velocity parallel to OX, OY and OZ (Fig. 281), and parallel to $O1$, $O2$ and $O3$. (OZ and $O3$ are perpendicular to the paper.)

Solution: The axis $O2$ is always in the plane of the shaft and rod; that is the coördinate frame $O123$ revolves about the vertical through O, at 30 r.p.m. Let r be radius of the sphere and R that of the circular track of the sphere on the cone as in Fig. 280. Then in one trip around the cone, the sphere turns over $2\pi R \div 2\pi r = R/r$ times. Hence the angular velocity of the sphere with respect to the frame $O123$ is $\frac{30R}{r} = 60$ r.p.m. OF and OG represent the angular velocities, 60 and 30 respectively; then OR (the diagonal of the parallelogram formed on OF and OG) represents the absolute angular velocity of the sphere. OR scales 75.7 r.p.m. (It is noteworthy that the instantaneous axis OR passes through the point of contact of sphere and cone as it should; for all points of the sphere, or its extension, on OR have no velocity and obviously the point of contact is such a point.)

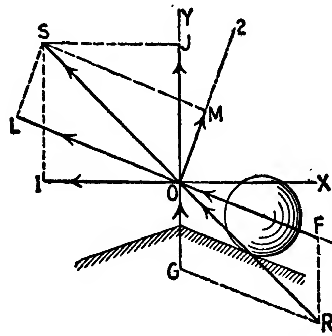


FIG. 281

For clearness we extended OR to OS (making $OS = OR$), and then resolved OS into components as required with results as follows:

$$\omega_x = OI = -56.4 \text{ r.p.m.}, \quad \omega_y = OJ = 50.5 \text{ r.p.m.}, \quad \omega_z = 0;$$
$$\omega_1 = OL = -70.1 \text{ r.p.m.}, \quad \omega_2 = OM = 28.2 \text{ r.p.m.}, \quad \omega_3 = 0.$$

149. Velocity of Any Point of the Moving Body. — Let P (Fig. 282) be any point of a moving body (not shown), fixed at O; x, y and z the (changing) coördinates of P with reference to fixed axes OX, OY and OZ; ω_x, ω_y and ω_z = the components of the angular velocity of the body with respect to those axes; v = the linear velocity of P; and v_x, v_y and v_z = the components of v along those axes. Then as will be proved presently

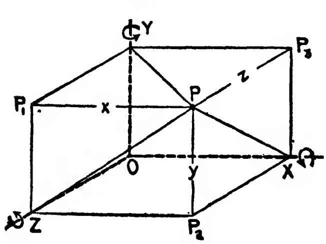


FIG. 282

$$v_x = z\omega_y - y\omega_z,$$
$$v_y = x\omega_z - z\omega_x,$$
$$v_z = y\omega_x - x\omega_y.$$

If the body were rotating about the x-axis only, then P would be describing a circle about X, and the velocity of P would be $XP \times \omega_x$. This velocity has no x component, and it is plain from the figure that the y and z components of that velocity respectively are $-z\omega_x$ and $y\omega_x$. These component velocities of P due to angular velocity ω_x are scheduled below; also the component velocities due to angular velocities ω_y and ω_z. It is plain from the schedule that the total component velocities due to the three angular velocities are as given by the above equations.

Rotation about OX produces $v_x = 0$, $v_y = -z\omega_x$, and $v_z = y\omega_x$.
Rotation about OY produces $v_x = z\omega_y$, $v_y = 0$, and $v_z = -x\omega_y$.
Rotation about OZ produces $v_x = -y\omega_z$, $v_y = x\omega_z$, and $v_z = 0$.

150. General Three-dimensional Motion Regarded as a Combined Translation and Rotation. — A rigid body can be displaced from one position A into another position B by means of a translation followed by a rotation. For, it is obvious that a translation can be selected so as to move any chosen point O of the body from its original position (in A) into its final position (in B). From this intermediate position, the body can be put into its final position by means of a rotation (of suitable amount) about a (certain) fixed axis through the final position of O (see Art. 146). The displacement might be effected in the reverse order, that is a rotation followed by a translation. For, a rotation about a fixed line through the "base point" O could be made so as to put the two lines OP and OQ, P and Q being two points of the body not in line with O, parallel to their final positions (in B); and a suitable translation would put those lines (and the body) into their final positions.

Evidently, the rotation and the translation could be made simultaneously. Therefore any actual motion of a body from one position into another may be regarded as a succession of infinitesimal simultaneous translations and rotations. All the translations may refer to the same base point, but in general the successive rotations do not occur about the same line of the body. Thus we may regard any motion of a rigid body as consisting of a translation (in which each point of the body moves just like the base point), combined with a rotation about a line through the base point, the line shifting about in the body, generally.

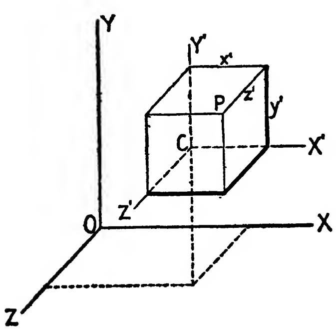


FIG. 283

151. Velocity of Any Point of the Body. — Let C (Fig. 283) be any base point and P any other point of the body (not shown). Also let $OXYZ$ be a fixed set of coördinate axes; $CX'Y'Z'$ another parallel set (moving with C); x', y' and z' the coördinates of P (as marked) relative to the second set; ω_x, ω_y and ω_z the axial components of the angular velocity of the body; $\bar{v}_x$, $\bar{v}_y$ and $\bar{v}_z$ the axial components of the (linear) velocity of the base point; and v_x, v_y and v_z the axial components of the velocity of P. Then as proved presently

$$v_x = \bar{v}_x + (z'\omega_y - y'\omega_z) \quad . \quad . \quad . \quad (1)$$
$$v_y = \bar{v}_y + (x'\omega_z - z'\omega_x) \quad . \quad . \quad . \quad (2)$$
$$v_z = \bar{v}_z + (y'\omega_x - x'\omega_y) \quad . \quad . \quad . \quad (3)$$

For, let v_x' be the x component of the velocity of P relative to C;

then (see Art. 155), $$v_x = \bar{v}_x + v_x',$$

and (see Art. 149) $$v_x' = z'\omega_x - y'\omega_z;$$ hence, etc.

Equations (2) and (3) could be deduced in a similar manner.

5. Relative Motion

152. Meaning of Path, Displacement, Velocity and Acceleration Relative to a Body. — Imagine a card lying on a page of this book and a bug on the card, and suppose that the bug can punch small holes rapidly through the card and through the page. Suppose also that the card is shifted about over the page whilst the bug walks and punches. The succession of holes in the card marks out the path of the bug relative to the card, and that in the page marks out its path relative to the page. Obviously the two paths would be very different, in general.

Again, imagine a running locomotive and consider the motion of a point of the rim of one of the wheels relative to that wheel, to the body of the locomotive, and to the earth. (i) With respect to the wheel the point does not change its position; it remains at rest. (ii) If a large drawing board were mounted on the locomotive body parallel to and near the inside face of the wheel so that a pencil fastened at the point on the rim of the wheel would trace a mark on the drawing board during the motion, then the mark thus made would be the path of the point relative to the locomotive. Clearly the mark would be a circle. (iii) If the board were supported on the earth in a vertical position and parallel to the track and so that the pencil would function as before, then the mark would be the path of the point relative to the earth. This (curved) path would be a succession of cycloids.

It is apparent now that a moving point may have quite different paths relative to different bodies of reference, and hence its motions (displacements, velocities and accelerations) relative to the bodies are different too. For convenience we shall call motions, paths, displacements and accelerations relative to the earth absolute, and motions, paths, etc., relative to any other body, relative. We are now ready to formulate the following definitions.

The displacement (for any interval) of a point relative to a body is the vector joining the initial and final positions of the point in the body (or in its extension), with sense from the initial to the final position. *The velocity of a point relative to a body* is the rate at which its relative displacement occurs relative to the body. *The acceleration of a point relative to a body* is the rate at which its relative velocity changes relative to the body.

To make clear the significance of these definitions we employ the bug-card-page illustration used above. Imagine C (Fig. 284) to be a card that slides about on the (stationary) page of the book while a bug (de-

noted in subscript by b) walks about on the card and by rapidly punching holes through it and the page, marks out its path relative to each. At a certain instant the card is in the position C; the bug is at M on the card and at F (under M) on the page. At the end of an interval Δt the card is in the position C'; the bug is at N on the card and at G (under N) on the page. The curved line MN on the card is the path of the bug relative to

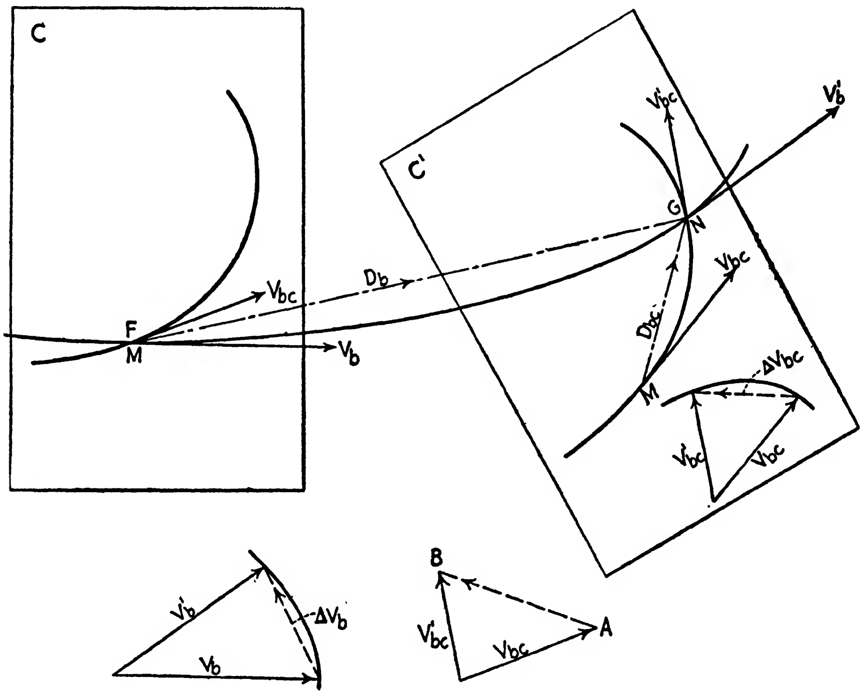


FIG. 284

the card and the curved line FG on the page is the absolute path of the bug. The vector MN on the card represents the relative displacement D_{bc} of the bug for the given interval, and the vector FG on the page represents its absolute displacement D_b.

The absolute velocity of the bug is $V_b = \lim D_b/\Delta t$; the relative velocity of the bug is $V_{bc} = \lim (D_{bc}/\Delta t)$. At the beginning of the interval the absolute velocity of the bug (represented by the vector V_b drawn on the page) is directed along the tangent to the absolute path at F and its magnitude is measured by the distance-on-the-page per unit time the bug is then moving as a result both of its walking and the sliding of the card; the relative velocity of the bug (represented by the vector V_{bc} drawn on the card) is directed along the tangent to the relative path at M and its magnitude is measured by the distance-on-the-card per unit time the bug is then walking. At the end of the interval the absolute velocity is as

represented by the vector V_b' (drawn on the page tangent to the absolute path at G); the relative velocity is as represented by the vector V_{bc}' (drawn on the card tangent to the relative path at N). The increment in absolute velocity is represented by the vector ΔV_b drawn on the page. The increment in relative velocity relative to the card is represented by the vector ΔV_{bc} drawn on the card; it is independent of the motion of the card. (The increment in relative velocity *relative to the page* is the vector difference between the initial vector V_{bc} and the final vector V_{bc}'; both being transferred to the *page*. Vector AB, Fig. 284, represents this increment. We do not call this quantity ΔV_{bc}; it is taken into account, when necessary, as shown in Art. 157.)

The relative acceleration of the bug is $A_{bc} = \lim (\Delta V_{bc}/\Delta t)$; it is the velocity relative to the card with which the point of the vector V_{bc} moves along the hodograph (drawn on the card) for the relative motion of the bug. The absolute acceleration of the bug is $A_b = \lim (\Delta V_b/\Delta t)$; it is the absolute velocity with which the point of the vector V_b moves along the hodograph (drawn on the page) for the absolute motion of the bug.

So long as only motion relative to a moving body is under consideration, the directions of relative displacement, velocity and acceleration are defined with reference to that body. Thus in considering the relative velocity and acceleration of the bug of the above illustration, the observer would imagine himself, so to speak, on the card, and his only standard of direction would be axes drawn on the card.

153. Motion Relative to a Point. — Thus far we have been discussing motion of a point relative to a body. By motion of a point relative to another point is meant the motion of the first point relative to a body which is fixed to the second point and free to move with it but without turning or rotating in any way. For example, if the card of the illustration of the preceding article is slid about on the page so that its edges remain fixed in direction, then the motion of the bug relative to the card is also its motion relative to any point of the card.

For another example, consider the motion of one boat relative to another, both regarded as mere points. Let a and b (Fig. 285) be the paths of boats or points A and B respectively with respect to the shore, and suppose that the points 0, 1, 2, etc. on the respective paths are the positions of A and B at 12, 1, 2, ... o'clock. We determine the path of A relative to B. For that purpose we first made the following table of coordinates of A and B relative to axes OX and OY on shore,

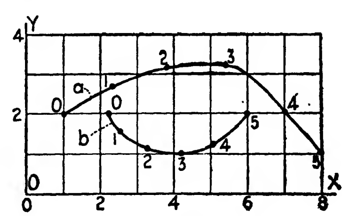


FIG. 285

read off from the figure. From this table we calculated, in an obvious manner, the cöordinates of A relative to coördinate axes parallel to OX

and OY with origin at B. These coördinates are called relative to B in the second table. The coördinates are plotted in Fig. 286; the smooth curve through the plotted points is the path of A relative to B.

COÖRDINATES RELATIVE TO OX AND OY, ON SHORE

Time		12	1	2	3	4	5
For A	$x =$	1.0	2.3	3.8	5.4	7.0	8.0
	$y =$	2.0	2.7	3.1	3.2	2.0	1.0
For B	$x =$	2.2	2.5	3.3	4.2	5.1	6.0
	$y =$	2.0	1.5	1.1	1.0	1.2	2.0

COÖRDINATES OF A RELATIVE TO B AND OF B RELATIVE TO A

Time		12	1	2	3	4	5
A rel. to B	$x =$	−1.2	−0.2	+0.5	+1.2	+1.9	+2.0
	$y =$	0.0	+1.2	+2.0	+2.2	+0.8	−1.0
B rel. to A	$x =$	+1.2	+0.2	−0.5	−1.2	−1.9	−2.0
	$y =$	0.0	−1.2	−2.0	−2.2	−0.8	+1.0

154. Motions of Two Points Relative to Each Other. — We introduce discussion of this subject by comparing the path of ship A (of the preceding article) relative to B (Fig. 286) with the path of ship B relative to A

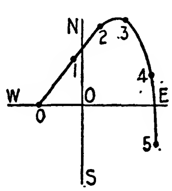


FIG. 286

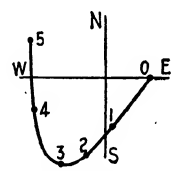


FIG. 287

(Fig. 287). The latter figure was constructed by plotting the coördinates of B with respect to A given in the second table of the preceding article. They were computed from the first table. It should be noted that the x coördinates (and the y) of A relative to B and those of B relative to A for any given time are opposite in sign.

The displacements of two points relative to each other for any interval of time are equal and opposite. Thus for the interval from 2 to 4 o'clock say, the displacement of A relative to B is the vector 24 of Fig. 286 and the displacement of B relative to A is vector 24 of Fig. 287, *apparently* equal and opposite vectors. To prove that they are equal and opposite, we need merely note from the last two tables that the x components of the displacements are equal and opposite (+1.4 and −1.4); also the y components (−1.2 and +1.2). The foregoing proof only covers relative motions in the same plane, but it could easily be extended to motions in space.

The velocities (and the accelerations) of two points relative to each other are equal and opposite at each instant. This proposition follows at once from the preceding one; for, since the displacements are equal and opposite for all intervals, the rates at which the displacements occur (that is, the relative velocities) are equal and opposite. And since the relative velocities are equal and opposite at each instant, their increments for any interval are equal and opposite, and hence the rates at which the velocities change (that is, the relative accelerations) are also equal and opposite at each instant.

155. Comparison of Motions of a Point Relative to a Moving and a Fixed Point. — (i) *The absolute displacement of a point P equals the displacement of P relative to point Q plus the absolute displacement of Q,* vectorial addition being understood; or

$$D_p = D_{pq} + D_q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To illustrate and prove this proposition, we use the bug-card-page combination of Art. 152. Let the bug be point P (of the proposition) and a corner of the card point Q. See Fig. 288; C_1 is the card and P_1 is the bug at the beginning of any interval of the motion; and C_2 is the card, and P_2 the bug at the end of the interval. The absolute displacement of P is P_1P_2 (also marked D_p), and the absolute displacement of Q is Q_1Q_2 (also marked D_q). At the end of the interval, the first hole punched in the card is at P'_1, determined by the parallelogram formed on Q_1P_1 and Q_1Q_2; hence the displacement of P relative to Q is P'_1P_2 (also marked D_{pq}). It is obvious from the figure that $D_p = D_{pq} + D_q$, which was to be proved.

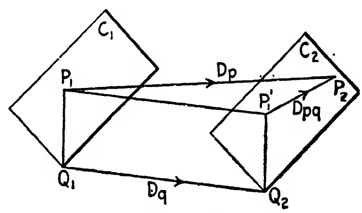


FIG. 288

(ii) *The absolute velocity (or acceleration) of P equals the velocity (or acceleration) of P relative to Q plus the absolute velocity (or acceleration) of Q,* vectorial addition being understood; or, for velocities

$$V_p = V_{pq} + V_q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This proposition as regards velocities follows at once from the preceding one as to displacements. Since the displacements are related as expressed by Eq. 1 for all intervals of time, the rates at which the displacements occur are similarly related. But these rates are the velocities respectively, hence Eq. (2).

To prove the proposition as to accelerations, we have the following relations pertaining to the beginning and end of any interval of time.

$$V'_p = V'_{pq} + V'_q, \quad \text{and} \quad V''_p = V''_{pq} + V''_q.$$

Subtracting the first equation from the second, term by term, gives

$$(V''_p - V'_p) = (V''_{pq} - V'_{pq}) + (V''_q - V'_q).$$

That is, the increment in the absolute velocity of P equals the increment in the velocity of P relative to Q plus the increment in the absolute velocity of Q, vectorial addition being understood. Since the increments in velocity are related as expressed by the last equation for all intervals of time, the rates at which the increments occur are similarly related, but these rates are the accelerations respectively; hence

$$A_p = A_{pq} + A_q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

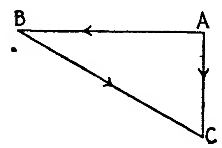


FIG. 289

EXAMPLE 1. A chauffeur driving at 25 miles in a rain storm without wind looks aside and observes the rain drops falling apparently at 60° from the vertical. It is required to determine the probable velocity of the rain drops.

Solution: The rain drop is P of the proposition above and the chauffeur is Q. AB (Fig. 289) is drawn to scale to represent the velocity of the car, AC in the direction of the absolute velocity of the rain drops, and BC in the direction of its velocity relative to the car. These lines determine C. And since $AB + BC = AC$, AC represents the required velocity or V_p (14.4 mi/hr).

EXAMPLE 2. In Fig. 290 OQ represents the crank and PQ the connecting rod of an engine. Their lengths respectively are $1\frac{1}{2}$ and 5 ft. It is required to determine the velocity and acceleration of P when the engine is running clockwise at 100 r.p.m. and the crank angle θ is 35°.

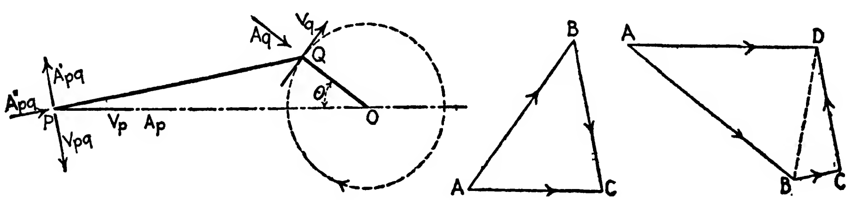


FIG. 290

Solution: Evidently the required velocity V_p and acceleration A_p of P are directed along the line PO. The velocity V_q of Q is directed as shown and equals $2\pi \times 1\frac{1}{2} \times 100 \div 60 = 15.71$ ft/sec., and the acceleration A_q of Q is directed as shown and equals $15.71^2 \div 1\frac{1}{2} = 164$ ft/sec/sec.

Since P is at a constant distance from Q, the path of P relative to Q is a circle (radius = 5 ft.), and the velocity of P relative to Q, V_{pq}, is perpendicular to the radius PQ as shown. To determine the required velocity, AB is drawn to scale to represent V_q, and from A and B lines parallel to V_p and V_{pq}. The intersection of these two lines determines C, and so vectorially,

$$AC = AB + BC.$$

Hence BC is V_{pq} (13.0 ft/sec.), and AC is V_p (11.1 ft/sec.).

The tangential and normal components of the acceleration of P relative to Q (or A_{pq}) are perpendicular to and along the radius PQ; the tangential component is marked A_{pq}' and the normal A_{pq}''. The latter is directed from P to Q as shown; its value is $13.0^2 \div 5 = 33.8$ ft/sec/sec. To determine the required acceleration, AB and BC respectively are drawn to scale to represent A_q (164) and A_{pq}''(33.8); then lines are

drawn through A and C parallel to A_p and A_{pq}'. Their intersection determines D, and so vectorially

$$AD = AB + BC + CD.$$

Hence AD represents A_p (153 ft/sec/sec.), and CD represents A_{pq}' (89 ft/sec/sec.). BD, the sum of BC and CD, represents A_{pq} (95 ft/sec/sec.)

156. Comparison of the Velocities of a Point Relative to a Moving and a Fixed Body. — *When a point P is moving relative to a moving body C then the absolute velocity of P equals the vector sum of its relative velocity and the absolute velocity of that point of C with which P coincides at the instant in question.*

Proof: For simplicity the proof is restricted to plane motion. Let C (Fig. 291) represent a card which moves about in the (fixed) plane of the page, and let the heavy curved line represent the path traced on C by a moving point P not shown. During a certain interval of time Δt the point P moves from M to N on the card, and the card moves from the position C to the position C' (the intermediate positions of the card are of no consequence). For the interval in question the relative displacement (on the card) of P is represented by the vector labeled D_{pc}; the absolute displacement (on the page) of M is represented by the vector labeled D_m, and the absolute displacement (on the page) of P is represented by the vector labeled D_p. Obviously

$$D_p = D_{pc} + D_m.$$

Therefore $$V_p = \lim \frac{D_p}{\Delta t} = \lim \frac{D_{pc}}{\Delta t} + \lim \frac{D_m}{\Delta t} = V_{pc} + V_m.$$

157. Comparison of the Accelerations of a Point Relative to a Moving and a Fixed Body. — *When a point P is moving relative to a moving body C then the absolute acceleration of P equals the vector sum of three accelerations, namely — the relative acceleration of P, the absolute acceleration of that point of C with which P coincides at the instant in question, and a so-called complementary acceleration.* The complementary acceleration equals twice the product of the relative velocity of P and the angular velocity of C at the instant in question; its direction is the same as that in which the point of the vector that represents the relative velocity of P is being made to move by the rotation of C, this vector being drawn from a fixed point.

Proof: As before, the proof is restricted to plane motion, the discussion being based upon the moving point P and the card C referred to in Art. 156. The absolute acceleration of P is $A_p = \lim \dfrac{\Delta V_p}{\Delta t} = \lim \dfrac{V'_p - V_p}{\Delta t}$, where V'_p and V_p are respectively the final and initial absolute velocities of P. The value of this limit will now be ascertained.

At the beginning of an interval P coincides with point M of the board and has an absolute velocity given by the vector sum of its relative velocity (V_{pc}) and the absolute velocity of $M(V_m)$. At the end of the in-

terval P coincides with point N of the board and has an absolute velocity given by the vector sum of its then relative velocity (V'_{pc}) and the absolute velocity of $N(V'_n)$. Therefore

$$V_p = V_{pc} + V_m, \quad \text{and}$$
$$V'_p = V'_{pc} + V'_n.$$

The values of V'_{pc} and V'_n will now be expressed in such form that $\lim \frac{V'_p - V_p}{\Delta t}$ can be evaluated.

The relative velocity of P at the end of the interval differs from its relative velocity at the beginning of the interval by (i) the amount of change relative to the card, which is ΔV_{pc}, and (ii) the additional amount of change relative to the *page* (due to the turning of the card through the angle $\Delta\theta$), which is equal to $2\, V_{pc} \sin \frac{1}{2}\, \Delta\theta$ (see Fig. 292). Therefore

$$V'_{pc} = V_{pc} + \Delta V_{pc} + 2\, V_{pc} \sin \tfrac{1}{2}\, \Delta\theta.$$

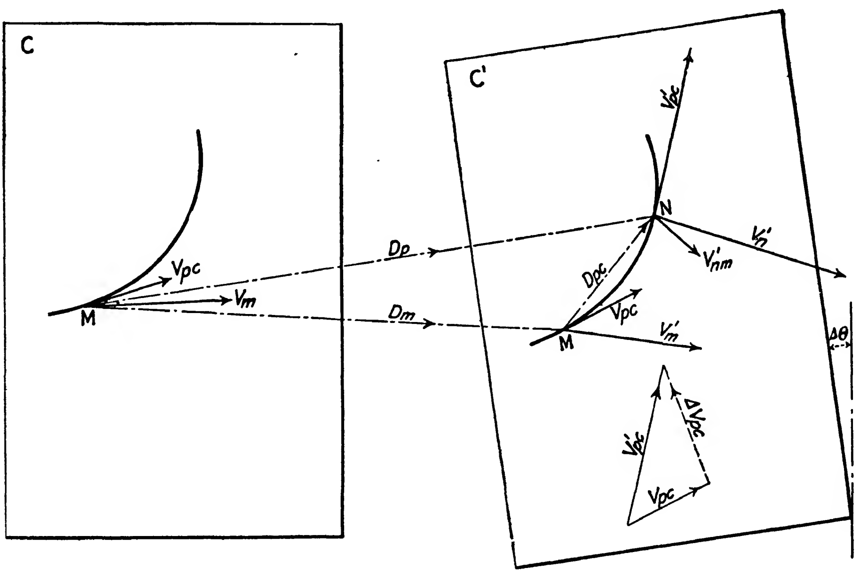


FIG. 291

The velocities and velocity changes mentioned are represented on Fig. 291 and on Fig. 292, the velocity scale of the latter figure being twice that of the former.

The absolute velocity of N at the end of the interval is given by $V'_n = V'_m + V'_{nm}$, where V'_m is the absolute velocity of M at the end of the interval and V'_{nm} is the velocity of N relative to M then. But $V'_m = V_m + \Delta V_m$ and $V'_{nm} = D_{pc}\omega'$, where ω' is the angular velocity of C at the end of the interval, and so

$$V'_n = V_m + \Delta V_m + D_{pc}\omega'.$$

The velocities and velocity changes mentioned are represented on Fig. 291 and Fig. 292. Summarizing the above, we have

$$V_p = V_{pc} + V_m; \text{ and}$$
$$V'_p = V_{pc} + \Delta V_{pc} + 2\, V_{pc} \sin \tfrac{1}{2}\, \Delta\theta + V_m + \Delta V_m + D_{pc}\omega'$$

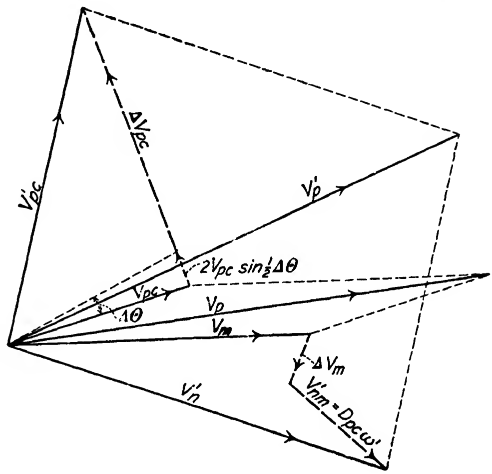


Fig. 292

whence

$$\Delta V_p = \Delta V_{pc} + 2\, V_{pc} \sin \tfrac{1}{2}\, \Delta\theta + \Delta V_m + D_{pc}\omega'.$$

Therefore

$$A_p = \lim \frac{\Delta V_p}{\Delta t} = \lim \frac{\Delta V_{pc}}{\Delta t} + 2 \lim \frac{V_{pc} \sin \frac{1}{2}\, \Delta\theta}{\Delta t} + \lim \frac{\Delta V_m}{\Delta t} + \lim \frac{D_{pc}\omega'}{\Delta t}.$$

But

$$\lim \frac{\Delta V_{pc}}{\Delta t} = A_{pc}; \quad 2 \lim \frac{V_{pc} \sin \frac{1}{2}\, \Delta\theta}{\Delta t} = V_{pc} \lim \frac{\Delta\theta}{\Delta t} = V_{pc}\omega;$$

$$\lim \frac{\Delta V_m}{\Delta t} = A_m; \quad \lim \frac{D_{pc}\omega'}{\Delta t} = \lim \frac{D_{pc}}{\Delta t} \lim \omega' = V_{pc}\omega, \text{ and so}$$

$$A_p = A_{pc} + A_m + 2\, V_{pc}\omega.$$

The limiting *direction* of the velocity change $2\, V_{pc} \sin \frac{1}{2}\, \Delta\theta$ is obviously the direction in which the point of the vector V_{pc} (Fig. 292) is being made to move by the rotation of C. The direction of the velocity change $D_{pc}\omega'$ is obviously perpendicular to the relative path of P at N; its *limiting* direction is perpendicular to this path at M, and this also is the direction in which the point of the vector V_{pc} (Fig. 292) is being made to move by the rotation of C. And so the complementary acceleration $2\, V_{pc}\omega$, which is the rate at which these combined changes occur, also has this direction.

EXAMPLE. Figure 293 represents a horizontal disc 4 ft. in diameter, in the surface of which there is a groove forming a circular track 1.5 ft. in diameter; the center of the track is 1 ft. from the center of the disc. The disc rotates counterclockwise (as viewed from above) about a vertical axis through its center O with an angular acceleration of 15 rad/sec/sec., while a small sphere travels uniformly clockwise around the grooved track, making 80 circuits per minute. When the disc (denoted by D) has attained an

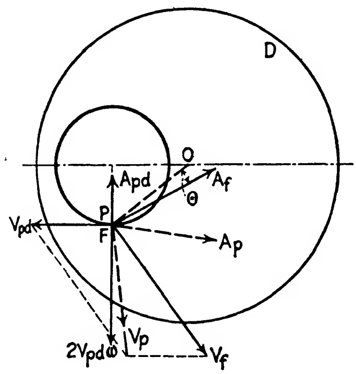


FIG. 293

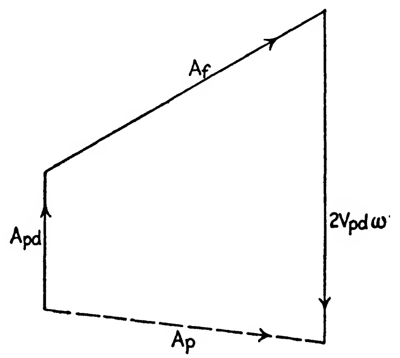


FIG. 294

angular velocity of 10 rad/sec., it and the sphere (regarded as a point and denoted by P) are in the positions shown. It is required to determine the absolute velocity and the absolute acceleration of the sphere at that instant.

Solution: Let F denote the point of the disc immediately under P. Then the absolute velocity of the sphere is given by

$$V_p = V_{pd} + V_f.$$

V_{pd}, the relative velocity of the sphere, is computed just as though the disc were stationary. Its magnitude is $(80 \times 1.5 \times \pi)/60 = 6.28$ ft/sec., and it is directed along the tangent to the groove as shown. V_f, the absolute velocity of F, has a magnitude equal to (distance FO) $\times\ \omega = 1.25 \times 10 = 12.5$ ft/sec. and is directed perpendicular to FO as shown. The vector sum of V_{pd} and V_f, determined graphically by means of the parallelogram in Fig. 293, is equal to 10.1 ft/sec. and is directed down and to the right at an angle of 81° to the horizontal as shown. This is V_p, the absolute velocity of the sphere.

The absolute acceleration of the sphere is given by

$$A_p = A_{pd} + A_f + 2\,V_{pd}\omega.$$

A_{pd}, the relative acceleration of the sphere, is computed just as though the disc were stationary; it consists simply of the normal acceleration of the sphere when traveling at a constant speed of 6.28 ft/sec. (its relative velocity) in a circle 1.5 ft. in diameter. Its magnitude is therefore $6.28^2/0.75 = 52.5$ ft/sec/sec. and it is directed toward the center of the track as shown. A_f, the absolute acceleration of F, has a tangential component ($r\alpha$) equal to $1.25 \times 15 = 18.75$ ft/sec/sec. and a normal component (v^2/r) equal to $12.5^2/1.25 = 125$ ft/sec/sec.; its magnitude is therefore $\sqrt{18.75^2 + 125^2} = 126.5$ ft/sec/sec. and it is directed up and to the right at an angle $\theta = \tan^{-1} 18.75/125 = 9°$ to the line FO, as shown. $2\,V_{pd}\omega$, the complementary acceleration of the disc, has a magnitude equal to $2 \times 6.28 \times 10 = 125.6$ ft/sec/sec. and is directed perpendicular

to V_{pd} as shown (i.e., it is directed like the motion that would be given the point of the vector V_{pd} by the rotation of the disc, if the vector V_{pd} were drawn from a fixed point). The component accelerations A_{pd}, A_f and $2\ V_{pd}\omega$ may be compounded either algebraically or graphically. The graphical solution is represented in Fig. 294 and shows that $A_p = 112$ ft/sec/sec. and is directed down and to the right at an angle of 7° to the horizontal, as shown.

KINETICS

CHAPTER X

FUNDAMENTAL FACTS; KINETICS OF A PARTICLE

158. Introductory. — In Kinematics (Chap. VIII and IX), we discussed various kinds of motion that a point or body may have, and the meaning and mutual relations of displacement, velocity and acceleration. We said nothing about the force or forces which, one knows, are always involved or associated with motions. The exact relations between the forces acting on a moving body and the motion itself are not apparent to the novice; they may indeed be obscure, and even as in the case of a gyroscope at apparent variance with what everyday experience might lead one to suppose. The study of these relations is the province of Kinetics. A knowledge of the principles of Kinetics enables one to determine how a given body moves when subjected to known forces, or to determine what forces are required to make a given body move in some particular way. The practical importance of the problem in engineering has been pointed out in the Introduction to this book.

159. Some Simple Examples of Forces Exerted on Moving Bodies. — (i) The simplest familiar case is that of a falling body. Two forces are acting on such a body, namely, gravity and air resistance or pressure. (ii) A ball being thrown is subjected to three forces, namely, gravity, air pressure, and the pressure of the thrower's hand. (ii) The thrown ball, after having left the hand, is subjected to two forces, namely, gravity and air pressure. (iv) A hockey puck after having been struck is subjected to three forces, namely, gravity, air pressure, and the "reaction" of the ice. (v) The connecting rod of a running engine is subjected to four forces; namely, gravity, air pressure, and the pressures exerted by the pins at its ends. (vi) A sled without load, being dragged in the usual way is subjected to four forces, namely, gravity, air pressure, the pull of the rope, and the reaction of the roadway, or ice field. (vii) A running automobile, with all its contents, is subjected to six (external) forces, namely, gravity, air pressure, and the reactions of the roadway on the four wheels. (The pressures of the exploding gas on the cylinder walls and pistons are internal forces; also the pressures between the meshing gears, "differential" parts, spring leaves, and all other such parts in contact.)

It should be noted that we are continuing here, as in Statics, to use the

word "force" as a general term for push or pull. Commonly, external forces are exerted on a moving body, as on a body at rest, by other bodies which are *actually in contact with it* (see preceding examples); but gravity and certain other forces are exceptions (see Art. 4, Statics). (See also Art. 170 for some erroneous uses of the word force.)

160. Brief Analysis of Some Simple Cases; Rectilinear Motion. — (1) A body falling from a moderate height falls with continually increasing speed. Because the (downward) speed increases, we infer that the resultant of the two forces acting on the body (gravity and air resistance) is directed downward; in other words, that the weight of the body is greater than the air resistance.

(2) The hockey puck in (iv) of the preceding article moves on the ice with decreasing speed. We ascribe this decrease to the resistances to which the puck is subjected, the air resistance and the frictional (horizontal) component of the reaction of the ice.

(3) Suppose, for simplicity, that the sled mentioned in (vi) of the preceding article is dragged on a level roadway, and that air resistance is negligible. Let W denote the weight of the sled, P the pull of the rope and R the reaction of the roadway. For simplicity of reasoning, we regard R as replaced by its vertical component N and its horizontal, or frictional, component F, and we will suppose that P is horizontal. Because the sled moves horizontally, we infer that N equals W, or that these two forces "balance." During the starting stage (getting up speed), P is greater than F. During the stopping stage (decreasing speed), F is greater than P. If, during the intermediate stage, the speed is constant, then P equals F, and all the forces on the sled are mutually balanced.

(4) Consider next the automobile mentioned in (vii) of the preceding article, and for simplicity suppose that it is running on a level roadway. Let W = weight of auto, P = air resistance, R' = reaction of roadway on both front wheels, and R'' = reaction on both rear wheels. Also let R_h' and R_h'' be the horizontal components, and R_v' and R_v'' the vertical components of R' and R'' respectively. Then $R_v' + R_v'' = W$; during a start, R_h'' acts forward and is greater than $R_h' + P$; during travel at constant speed, R_h'' acts forward and $R_h'' = R_h' + P$; and during a stop R_h'' acts backward.

Inferences. — It appears from the preceding illustrations, that when a body has rectilinear motion then: If the speed is constant, the external forces acting on the body are mutually balanced, or their resultant is nil. If the speed is varying, then those forces are not balanced; if the speed is increasing then their resultant acts in the direction of the motion, and if the speed is decreasing, it acts opposite to the direction of motion.

161. Brief Analysis of Some Simple Cases — Curvilinear Motion. — (1) Imagine a hockey puck moving over the ice and a player overtaking and intending to " shoot " it toward his left. What will he do? He will

strike the right-hand side of the puck, of course. Or if he wishes to thus deflect and "nurse" the puck, he will continually press moderately on the rear right side. This pressure P has a component along the tangent to the path of the puck and one along the normal. The latter changes the direction of the velocity; the resultant of the former and the friction (on the puck) changes the speed.

(2) Figure 295 represents a smooth ball fitting loosely in a box which can be rotated about the vertical axis YY. The ball can touch only two sides and the bottom of the box at any one time. Hence there are not more than four forces acting on the ball, namely, gravity W, the reaction N of the bottom, and the pressures of the two sides. N equals or balances W. Whether the system is rotated clockwise or counterclockwise with increasing, decreasing, or uniform speed, the ball presses against the outer side A, as we all know from experience, and hence the side A presses inward (toward the axis of rotation) on the ball; let P denote this inward force. If the rotation is non-uniform, the ball presses against B or C as we all know, and B or C presses against the ball; let Q denote this force

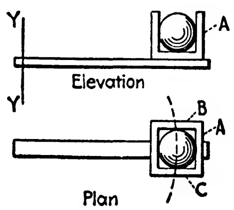


FIG. 295

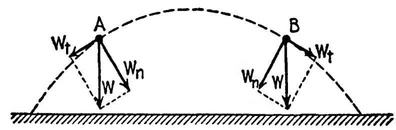


FIG. 296

on the ball; P deflects the ball, that is, changes the direction of its motion or velocity; Q increases or decreases the speed.[1]

(3) Consider the motion of a thrown base ball: — Suppose that it was thrown without "spin" so that it does not "curve"; and, for simplicity, let us neglect air resistance. Then we say it moves under the influence of gravity only and the initial velocity. In Fig. 296 we show the ball in two positions, A (ball rising) and B (ball falling), the directions of its velocity there, gravity W, and the components of W along the tangent and the normal to the path. At A, the component W_t is decreasing the speed and at B it is increasing the speed. W_n does not affect the speed but changes the direction of the motion of the ball continually.

Inferences. — It appears from the foregoing that whenever a (small) body has a curvilinear motion, then the external forces acting on it are not balanced. There is an unbalanced (deflecting) force or component of force directed along the normal to the path (of the center of gravity) toward the

[1] In analysis of similar cases some would use the terms "centripetal" and "centrifugal" force. The (inward) force P on the ball is the centripetal force, and the (equal) outward force on the box is the centrifugal force. Careless use of these terms sometimes results in confusion through failure to note that these forces do not act on the same body.

center of curvature. If the speed is changing, then there is also an unbalanced force or component of force along the tangent to the path; it is this force or component which increases or decreases the speed, as the case may be.

162. Some Experimental Facts Concerning Motion; Effects of Forces. — We will now briefly describe some experiments and state the results which *would* be obtained if the apparatus were quite perfect, all in answer to questions (1), (2) and (3) below. We believe such apparatus *could* be made to work satisfactorily but it would be quite complicated. Simpler apparatus is in successful use, but the one described below suits our present purpose better.

Figure 297 represents a "body" which can be made to travel on the horizontal floor or track by means obvious in the figure. It consists of a light, easy-moving cart, a spring scale for indicating the pull of the cord, a speedometer for indicating the speed of the cart, and any suitable load. We will assume that the spring scale and the speedometer are so arranged as to record their indications on a single sheet of paper suitably moved by clock-work. Then the spring scale gives a force-time and the speedometer a speed-time graph. (The spring scale, speedometer, etc., are not shown in the figure; they are to be imagined within the housing shown.)

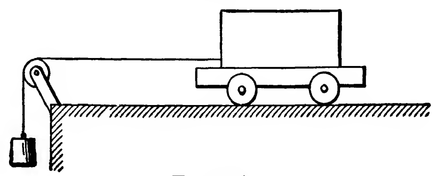


FIG. 297

Practically all of the described body has a motion of translation. The external forces on it consist of gravity (earth-pull or weight), the pull of the cord, and the reaction of the track. The vertical component of the reaction balances the weight; the horizontal component could be made small and negligible compared with the pull. Hence, practically, the pull is the resultant or unbalanced force on the body.

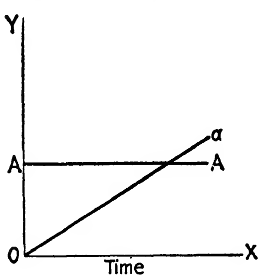


FIG. 298

(1) How does a constant force affect the (rectilinear) velocity of a body? — Imagine that a trial or experiment has been made with apparatus described. Figure 298 is the record. AA was drawn by the spring-scale indicator, and Oa by the speedometer indicator. Ordinates to AA from OX (or zero line) represent the pull of the cord, and ordinates to Oa (from OX) the speed. Oa, being straight, shows that the velocity increased uniformly, or that the acceleration was constant. Indeed, if a recording accelerometer also were included in the apparatus, it would trace a horizontal line. Other trials with different suspended

bodies would show, in each case, constant pull and uniformly varying velocity, or *constant acceleration.*

(2) How do unequal constant forces affect the same body? — Figure 299 is the record of an experiment of several trials with different suspended bodies. *AA* and *Oa* might be the record for one trial; *BB* and *Ob* that for another, etc. As expected, the larger the pull, the larger the *rate of increase of the velocity* (slope of velocity-time graph). Careful comparison of these rates (accelerations) respectively with the pulls would show them to be *exactly proportional to the pulls.*

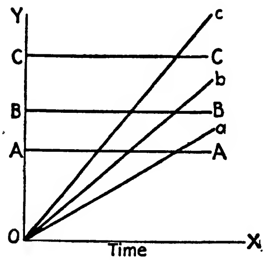


FIG. 299

(3) How do equal constant forces affect different bodies? — Imagine several trials with the apparatus, a different body on the cart in each trial and suspended bodies chosen so that the spring-scale readings in all trials are equal. The acceleration in each trial would be constant, but as expected the several values of the acceleration would be different. It would be least for the greatest load and greatest for the smallest load. (By load we mean the entire amount of material dragged by the cord.) Careful comparisons would show the accelerations respectively to be *inversely proportional to the weights of material moved.*

163. Weight and Mass. — See Art. 3 for our first remarks on weight. The reader should gather that we, following scientific usage, mean by weight of a body, the Earth's pull or attraction for it; and that weight of a body, so defined, varies slightly from place to place. (A spring scale, if sensitive enough, would detect this variation in weight; a beam scale of course would not.)

Material is measured in various ways; generally by "bulking" or by weighing. Weighing a given body by beam scale gives the same results at various places, and this is in accordance with one's natural notions about material. It is convenient to have a name for amount of material in a given thing determined in this particular manner; the word *mass* is often used for this purpose and we follow that usage. Masses of bodies are proportional to their weights *at the same place.*

It will be noticed that we are avoiding the double meaning of the common usage of the word weight (as explained in footnote under Art. 3) reserving the word for one of its common meanings, and using mass for the other.

Units of Mass. — The *pound* and the *kilogram* (commonly called standards of weight) are the principal standards of mass. They are certain pieces of metal preserved in London and Paris respectively. There are many familiar units of mass based on these standards, the ounce, ton, etc. See Art. 166 for still others.

164. Force-acceleration Equation, First Form. — Imagine a body falling freely and without air resistance. During the fall it moves under the ac-

tion of gravity only (the weight W of the body) which produces the acceleration g. Next imagine it subjected to any single constant force F and let a denote the acceleration produced. Then in accordance with the conclusion of (2) of Art. 162

$$\frac{a}{g} = \frac{F}{W}, \quad \text{or} \quad a = \frac{F}{W}g, \quad \text{or} \quad F = \frac{W}{g}a.$$

These we shall call force-acceleration equations because they express precisely the relation between F and a. W and g are generally known; hence the equations enable one to calculate the acceleration produced by a given force F, or the force F required to produce a given acceleration.

Units for Use in the Foregoing Equation. — Any unit of force may be used for F and W, and any unit for g and a. When a gravitational unit of force is used — such are convenient in most engineering calculations — then, strictly, the numerical value of g used should correspond to the "locality" of the unit-force used. That is, when one is about to make a calculation by means of the foregoing equation, implying the New York pound-force say, then he should use for g its value for New York. As already stated, the variation in g is negligible in most engineering calculations, and we generally use 32.2 feet per second per second or even 32 for simplicity.

165. Force-acceleration Equation, Second Form. — Imagine any number of different bodies of mass M', M'', etc. subjected respectively to single forces F', F'', etc.; let a', a'', etc. denote the accelerations respectively. According to (2) and (3) of Art. 162, the accelerations are directly proportional to the forces and inversely proportional to the masses; that is

$$a' : a'' : \cdots\cdots = \frac{F'}{M'} : \frac{F''}{M''} : \cdots\cdots$$

or

$$\frac{F'}{M'a'} = \frac{F''}{M''a''} = \quad \text{etc.}$$

Thus it appears that these ratios have a common value, say K; and hence, each and every one of them can be rewritten

$$F = KMa, \quad \text{or} \quad a = F/KM \quad \ldots\ldots\ldots \quad (1)$$

where a denotes the acceleration produced by any force F acting singly on a body of mass M and K is the constant as yet undetermined.

The value of the constant can be derived from any experiment in which F and a can be determined. One needs only to substitute the determined values and the value of M in the preceding equation, and then solve for K. The numerical value of K thus derived depends on the *units* used for expressing the values of F, a and M. Inasmuch as nearly all equations of dynamics depend on the preceding one, simplicity is gained if these units are chosen so as to make $K = 1$. Such choice is commonly made (see following article), and then

$$F = Ma \quad \text{or} \quad a = F/M \quad \ldots\ldots\ldots\ldots \quad (2)$$

166. Kinetic Systems of Units. — Units chosen so as to make K of Eq. (1) of the preceding article equal to 1, together with others based systematically on them, are called a *kinetic system of units.* It follows from this definition and Eq. (2) of the preceding article that "unit force gives unit mass unit acceleration." To insure this relation three of the four units of force, mass, length and time may be selected at pleasure, but the fourth must be specially chosen (see below). The three are called *fundamental units of the system;* the fourth (and others based on them) *derived.*

Absolute Kinetic Systems. — In these, all fundamental and derived units are absolutely constant in value, being independent of locality especially. Units of length, mass, and time are the fundamental units.

Centimeter-Gram-Second (C. G. S.) System. — The units of length, mass, and time are the centimeter, the gram (one-thousandth of a kilogram, see Art. 3), and the second respectively. The corresponding unit of force, that is, the force which acting on a gram mass for one second gives it a velocity of one centimeter per second, is called a *dyne.* This system is now universally used in theoretical physics. (See Appendix A.)

Foot-Pound-Second (F. P. S.) System. — The units of length, mass, and time are the foot, the pound (see Art. 3), and the second respectively. The corresponding unit of force, that is, the force which acting on a pound mass for one second gives it a velocity of one foot per second, is called a *poundal.* This system has never met with favor; it is not used herein, but is mentioned and explained because of the relations which it bears to other important systems. (See Appendix A.)

Gravitational Kinetic Systems. — In these, some of the units depend on gravity, whence the name. Units of length, force and time are the fundamental units.

Foot-Pound (force) -Second System. — The units of length, force and time are the foot, the pound (see Art. 3), and the second respectively. The corresponding unit of mass, that is, one in which a pound force would produce in one second a velocity of one foot-per-second, is called a slug.[1] (See Appendix A.)

Meter-Kilogram (force) -Second System. — The units of length, force, and time are the meter, the kilogram (see Art. 3), and the second respectively. The corresponding units of mass, that is, one in which a kilogram force would produce in one second a velocity of one meter-per-second, is called herein a metric slug. (See Appendix A.)

Relation Between Mass and Weight of a body numerically expressed in

[1] This is a relatively new term, and is not in general use. It was proposed by Professor A. M. Worthington in his *Dynamics of Rotation;* he took it from the word sluggish. Other names have been proposed for this unit; for example, "matt," from matter; "ert," from inertia; "geepound," 32.2 pounds; and perhaps others. But only slug has any vogue; it is used freely in some literature on Aerodynamics (see for example the reports of the Advisory Committees for Aeronautics of Great Britain, and of the United States).

kinetic units. — Imagine a body falling freely, that is without air resistance. If to this motion we apply Eq. (2) of the preceding article, F is gravity, or the weight W of the body, and a is the acceleration g "due to gravity"; hence

$$W = Mg \quad \text{or} \quad M = W/g.$$

167. Particle. — In the foregoing articles we have explained how forces affect the rectilinear motion of bodies. In the next chapter we make such explanations with reference to any sort of motion. Preliminary to these explanations we consider next the laws of motion of a particle (see next article).

By "particle" is meant a body, the dimensions of which are negligible in comparison with the range of its motion. In any motion of a particle no distinction need be made between the displacements, velocities, or accelerations of different points of the particle, for they are equal or practically so. By displacement, velocity, or acceleration of the particle is meant the displacement, velocity, or acceleration of any point of the particle.

168. Laws of Motion of a Particle. — Law 1. *When no force is exerted on a particle, then it remains at rest or moves in a straight line with constant speed.* These two states, rest or uniform rectilinear motion, may be regarded as *natural states* for a particle; and (see law 2) a state of curvilinear motion or nonuniform rectilinear motion as *unnatural*, forced or imposed on the particle by a push or pull.

Law 2. *When a single force is exerted on a particle then it is accelerated; the direction of the acceleration is the same as the direction of the force, and its magnitude is proportional to the force directly and to the mass of the particle inversely.* It should be noted that the law is not restricted to cases of rectilinear motion. If, for instance, a particle is given an initial vertical velocity somehow and then subjected to a single horizontal force, then obviously the particle will describe a curved path; and according to the law the acceleration a and the force F agree in direction at each instant of the motion. Also, if m denotes the mass of the particle, $a \propto F/m$. Or, if kinetic units (Art. 166) be used for a, F and m, then

$$a = F/m, \quad \text{or} \quad F = ma.$$

Law 3. *When one particle exerts a force upon another, then the latter exerts one on the former, and these two forces are equal, colinear, and opposite.* This law has already been discussed (Art. 7).

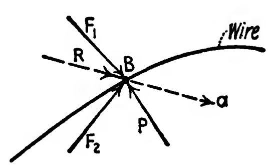


FIG. 300

169. Particle Under the Action of Several Forces. — Let B (Fig. 300) represent a bead on a wire, and imagine the bead to be sliding along the wire under the action of several applied forces F_1 and F_2, gravity W, and the reaction P of the wire. In any particular position, the bead has a

definite velocity v[1] and acceleration a.[1] Obviously some single force R acting alone would give the particle that identical acceleration. According to law 2 (Art. 168) R would have to act in the direction of a and have a value equal to ma, m denoting mass of the bead. This force R is the resultant (see Art. 15) of F_1, F_2, P and W. For, R reversed and acting alone on the bead would give it an acceleration $-a$; and hence R reversed and acting *with* the other forces would produce zero acceleration, that is R reversed would "balance" the other forces. Therefore R reversed is the equilibrant or anti-resultant of the forces, and R is their resultant.

All the relations between concurrent forces and their resultant developed in Statics hold here also for F_1, F_2, etc., and R. In particular, the component of R along any line equals the algebraic sum of the components of the forces F_1, F_2, etc. along the same line (Art. 26), or if we call the line an x-axis, then $R_x = \Sigma F_x$.

Let α = the angle between the direction of the acceleration a and any line which we will call the x-axis; then since $R = ma$,

$$R \cos \alpha = ma \cos \alpha, \quad \text{or} \quad R_x = ma_x;$$

also since $R_x = \Sigma F_x$,

$$\Sigma F_x = ma_x.$$

170. Inertia; Erroneous Notions. — The word inertia is sometimes used to refer to the behavior of matter as described in the laws of motion: To refer to the fact that the *natural state* of a particle is rest or uniform rectilinear motion (law 1); that a particle inclines, so to speak, to remain in the natural state, departing therefrom only on account of an outside influence, namely, external force (law 2); and that it is so reluctant to change that state that it always resists or exerts a force on the body which causes the change (law 3). In this sense all particles are equally inert or have the property of inertia in the same full degree.

The word is used also quantitatively in reference to the second law. Thus it is said that different particles require unequal forces to overcome their inertias, and that they have different amounts of inertia. Such statements really imply a mode of measuring or expressing amounts of inertia. A very natural way would be to take as measure the force-per-unit-acceleration required to give the particle (or body) rectilinear acceleration. Thus if a denotes the acceleration given a particle (or body) by a force F, then the inertia of the particle is F/a. Or, since F may be any force, it might be gravity, or the weight W of the particle producing an acceleration g; then W/g also would be the value of its inertia.

[1] The use of and V and A to denote, as in Kinematics, (vector) velocity and acceleration, is discontinued at this place, because no confusion will result if v (and a) be used to denote either vector velocity (and acceleration) or, as in Kinematics, the magnitude only of the vectors.

Still another usage, generally vague and often erroneous, is "force of inertia." For example, concerning the motion of a hockey puck projected along the surface of smooth ice, it is stated sometimes that the puck is urged onward by the (or its) force of inertia. This statement is erroneous; the only forces acting on the puck, after projection, are gravity, air pressure, and the reaction of the ice; there is no force urging the puck onward; the puck moves onward — for a time — because it *was* (forcibly) projected, and *in spite* of the retarding influence of the air and the ice. Were it not for this influence, the puck would move across the entire field at constant velocity, naturally, not because of any force urging it onward but because of the absence of any force to change its natural state (of uniform rectilinear motion).

Again, concerning the difficulty of keeping one's balance when standing in a street car that starts or stops suddenly, or turns a corner, it is said sometimes that one is thrown backward, forward or outward by the force of inertia. Such statements are wrong. Here is a correct one. In a start one would naturally remain at rest; the starting force is applied to the feet; the lower part of the body gets started first; the upper part lags behind until one stiffens up and recovers balance. Similarly for a stop. In approaching the corner or curve at constant speed, say, along a straight track, one has a natural state of uniform rectilinear motion which the body would naturally maintain. The force which swerves the body from the straight path is applied at the feet; the lower part of the body is swerved first; the upper part lags behind until one stiffens up and recovers balance.

"But," some one may say, "I have felt the force of inertia when starting or stopping a heavy body, and also when whirling a heavy body at the end of a rope somewhat as a hammer thrower swings his hammer prior to the actual throw." These statements are not wrong, if they refer to the force exerted by the heavy body on the person, which is really what is felt. But, from such experience, some are apt to infer that the force felt "is due" to an equal but opposite force on the heavy body which they also call the force of inertia. This is wrong; the only forces exerted on, say, the *whirling* body are gravity and the inward pull of the rope.

CHAPTER XI

KINETICS OF A BODY

§ 1. Motion of the Mass-center of Any Body.

171. Preliminary. — The equation $\Sigma F_x = ma_x$, given in Art. 169, completely defines the relation existing between the motion of a *particle* and the forces that act on it. With this as a basis, we shall now develop equations that define the relations between the motion of the mass-center of any body and the external forces that act on the body.

172. Equations of Motion of the Mass-center. — Let P' (Fig. 301) represent a particle of a body (not shown) having any motion. Consider any line as x-axis, and let this line be one of a rectangular set of axes OX, OY and OZ. Also let x', y' and z' be the (varying) coördinates of this particle, m' its mass, a' its acceleration, a_x' the x component of a', and $\Sigma F_x'$ the x component of the system of forces acting on m'. Then $\Sigma F_x' = m'a_x'$. Likewise for any other particle P'' of the body we may write $\Sigma F_x'' = m''a_x''$. Suppose now that similar equations have been written down for all the particles that compose the body. The sum of the left-hand members of all such equations equals the sum of the right-hand members, that is

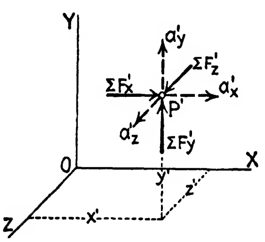


Fig. 301

$$\left.\begin{array}{l}\text{Summation of the components along}\\ \text{any axis of all forces acting on all par-}\\ \text{ticles}\end{array}\right\} = m'a_x' + m''a_x'' + \text{etc.}$$

The foregoing equation can be greatly simplified, as will now be shown. The left-hand member includes components of all forces, external and internal, that act on the body. Since the internal forces occur in pairs of equal and opposite forces their components cancel in the summation. It follows that the value of the left-hand member can be obtained by summing the components of the *external forces only.*

The right-hand member of the equation is equal to $M\bar{a}_x$, where M is the mass of the body and $\bar{a}_x$ is the x component of the acceleration of its mass-center; proof follows. From Art. 89, the moment of the weight of a body with respect to any axis is equal to the sum of the moments of the weights of its parts, or

$$w'x' + w''x'' + \cdots \quad \cdots\cdots = W\bar{x}.$$

But the mass of any part is proportional to its weight, and so we may write

$$m'x' + m''x'' + \cdots\cdots = M\bar{x}.$$

Differentiating both sides of this equation with respect to time, and noting that $dx'/dt = v_x'$ etc., we have

$$m'v_x' + m''v_x'' + \cdots\cdots = M\bar{v}_x.$$

Differentiating again and noting that $dv_x'/dt = a_x'$, etc., we have

$$m'a_x' + m''a_x'' + \cdots\cdots = M\bar{a}_x.$$

It is thus seen that the component of the external system of forces along OX is equal to the product of the mass of the body and the x component of the acceleration of its mass-center. Since the axis OX represents *any* line, it follows that any number of equations of the form $\Sigma F_x = M\bar{a}_x$ may be written for a body, one for each direction that may be chosen. Only three such equations would be independent. These might be written for any three axes no two of which were parallel but it is convenient to choose rectangular axes and to write

$$\Sigma F_x = M\bar{a}_x, \qquad \Sigma F_y = M\bar{a}_y, \qquad \Sigma F_z = M\bar{a}_z.$$

These are the equations of motion of the mass-center.

173. Principle of Motion of the Mass-center. — The equations of motion of the mass-center constitute a mathematical statement of an important principle which we shall call the principle of the motion of the mass-center. It may be put into words as follows: *In any motion of a body (whether rigid or not), the algebraic sum of the components (along any line) of all the external forces equals the product of the mass of the body and the component of the acceleration of the mass-center along that line.* It appears, then, that the motion of the mass-center of a body is the same as that of a particle of mass equal to that of the body, in the position of the mass-center of the body, and acted on by forces equal to and parallel to the external forces exerted on the body.

The exact meaning and limitations of the principle of the motion of the mass-center must be understood. It completely describes the relations existing between the external forces that act on a body and the acceleration of the mass-center of the body. It shows that for forces of given magnitude and direction the mass-center of a body moves the same whether the body turns or has motion of translation; whether the body is rigid, like a steel bar, or flexible, like a chain, or fluid, like a quantity of water. And it shows that neither the place of application nor the position of the line of action of a force is material, so far as the motion of the mass-center is concerned. But the principle tells us nothing whatever concerning the motion of points of the body other than the mass-center.

174. Typical Problems; Examples. — The most common application of the principle of the motion of the mass-center is to the solution of two types of problems which may be stated, and solved, as follows:

Typical Problem 1. — A given body is moving in a certain way, the path, velocity and acceleration of its mass-center being known or easily determinable. Other bodies, either at rest or in motion, exert forces on the body in question. Some of these forces are wholly known; some are partially or wholly unknown. Some or all of the unknown quantities (magnitude, line of action, or sense) are to be determined.

Solution: (i) Construct a free body diagram (Art. 54) for the body in question. (ii) Represent on this diagram the path of the mass-center of the body, and the acceleration, or components of the acceleration, of the mass-center. (iii) Apply the equations $\Sigma F_x = M\bar{a}_x$, etc., to the body and solve for the quantities that are to be determined.

The free body diagram is constructed exactly as in Statics; it consists of a sketch of the body, with the external forces that act on it represented as fully as possible. The path of the mass-center may be represented by a dotted line, any convenient point of view being assumed. In simple cases of rectilinear motion of translation it is not necessary to represent the path. The acceleration, or components of acceleration, of the mass-center should be represented by broken lines (with arrow heads) in order to avoid the possibility of confusing accelerations and forces. In simple cases of rectilinear motion, where the direction of the acceleration is obvious, it is not necessary to represent it. The axes along which the forces are summed up should for convenience be selected with regard to the circumstances of the problem. In cases of rectilinear motion, it is usually convenient to resolve parallel to and perpendicular to the direction of motion; in cases of curvilinear motion, it is usually convenient to resolve along the tangent to the path, the (principal) normal, and at right angles to the plane of these two.

Typical Problem 2. — A given body is acted on by certain forces, these being known or easily determinable. Something concerning the motion of the mass-center of the body (path, velocity or acceleration) is to be determined.

Solution: (i) Construct a free body diagram for the body in question. (ii) Apply the equations $\Sigma F_x = M\bar{a}_x$, etc. to the body and solve for the quantities that are to be determined.

In problems of either type that are at all complicated it is well to formulate a general plan of solution — to figure out the successive steps required and see the way through to the desired result — before commencing the actual computations. Examples 2 and 3 below illustrate this point.

EXAMPLE 1. A block weighing 50 lbs. rests on a horizontal floor; the coefficient of kinetic friction between the block and the floor is 0.3. A horizontal force of 30 lbs. is applied to the block. It is required to determine the resulting acceleration.

Solution: The free body diagram of the block is shown in Fig. 302; the forces acting are its weight, the applied force of 30 lbs., and the reaction of the floor, consisting of the normal component N and the friction component, the latter being equal to 0.3 N since slipping occurs. The direction of the acceleration is horizontal and to the right, therefore

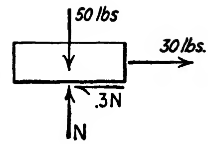


FIG. 302

$$\Sigma F_y = -50 + N = 0, \text{ whence } N = 50 \text{ lb., and } 0.3\,N = 15 \text{ lbs.; and}$$

$$\Sigma F_x = 30 - 15 = \frac{50}{32.2}\,a, \text{ whence } a = 9.66 \text{ ft/sec/sec.}$$

EXAMPLE 2. Figure 303 represents two blocks A and B, which rest on a smooth horizontal floor and are coupled together by a cord which is inclined at 30° to the horizontal. A weighs 100 lbs.; B weighs 60 lbs. The coupling cord can sustain a tension of 25 lbs. It is required to determine how large a horizontal force P, applied to block A as shown, must be to break the cord.

Solution: The plan of solution is as follows: Assume 25 lb. tension in the cord and ascertain the acceleration this would give block B, then determine what force P would give this acceleration to the entire system; any greater force would break the cord.

The free body diagram for B is shown in Fig. 304; the acceleration is horizontal and to the right, therefore

$$\Sigma F_x = 25 \cos 30° = \frac{60}{32.2}\,a, \text{ whence } a = 11.6 \text{ ft/sec/sec.}$$

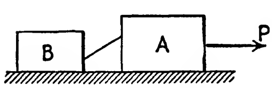


FIG. 303

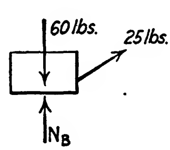


FIG. 304

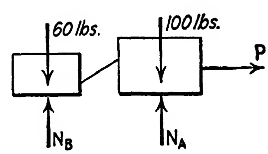


FIG. 305

The free body diagram for the entire system is shown in Fig. 305; the acceleration is to the right and equal to 11.6 ft/sec/sec., therefore

$$\Sigma F_x = P = \frac{160}{32.2}\,11.6, \text{ whence } P = 57.6 \text{ lbs.}$$

Any greater force than 57.6 lbs., applied as P, would cause the cord to break.

EXAMPLE 3. Against the vertical face of a block A which rests on a horizontal floor is placed a second block B (Fig. 306). A weighs 150 lbs.; B weighs 60 lbs.; the coefficient of friction between A and the floor is 0.1 and between A and B it is 0.5. It is required to determine what horizontal force P applied to A as shown will prevent B from slipping down.

Solution: The plan of solution is as follows: Ascertain what normal pressure between A and B is required to develop sufficient friction to prevent B from slipping down, determine the acceleration that would be given B by this pressure, then find what force P would be required to give the entire system this acceleration.

The free body diagram for B, assumed just on the point of slipping down, is shown in Fig. 307. The acceleration of B is to the right, therefore

$$\Sigma F_y = -60 + 0.5\,N = 0, \text{ whence } N = 120 \text{ lbs.; and}$$

$$\Sigma F_x = 120 = \frac{60}{32.2}\,a, \text{ whence } a = 64.4 \text{ ft/sec/sec.}$$

The free body diagram for the entire system is shown in Fig. 308. The acceleration is to the right and equal to 64.4 ft/sec/sec., therefore

$$\Sigma F_y = -210 + N = 0, \text{ whence } N = 210 \text{ lbs. and } 0.1\,N = 21 \text{ lbs.; and}$$

$$\Sigma F_x = P - 21 = \frac{210}{32.2}\,64.4, \text{ whence } P = 441 \text{ lbs.}$$

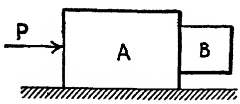


FIG. 306

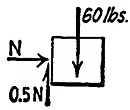


FIG. 307

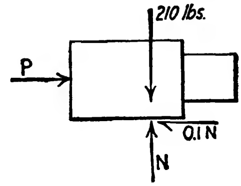


FIG. 308

Any force equal to or greater than 441 lbs., applied as P, would prevent B from slipping down.

EXAMPLE 4. A sphere C (Fig. 309) is laid in a box which is mounted on a board as shown, and the whole system is then rotated about a vertical axis AB. The weight of the sphere is 30 lbs., $AC = 2$ ft., and the rate of rotation (uniform) = 60 rev/min. It is required to determine the pressures exerted by the box on the sphere.

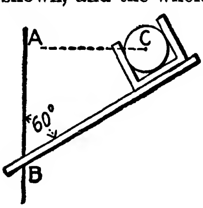


FIG. 309

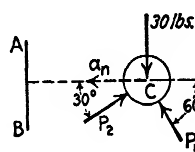


FIG. 310

Solution: The free body diagram for the sphere is shown in Fig. 310. The forces acting are the weight of the sphere, the pressure P_1 of the bottom of the box, and a pressure P_2 exerted by one of the ends. It is *assumed* that the sphere rests against the lower end of the box; the complete solution will reveal whether this assumption is correct. The path of the mass-center is a horizontal circle 4 ft. in diameter; the velocity of the mass-center is $(\pi \times 4 \times 60)/60 = 12.5$ ft/sec.; the components of the acceleration of the mass-center are therefore $\bar{a}_n = 12.5^2/2 = 78$ ft/sec/sec., $\bar{a}_t = 0$, and $\bar{a}_y = 0$. The following equations apply:

$$\Sigma F_t = 0,$$

$$\Sigma F_y = -30 + P_1 \sin 60° + P_2 \sin 30° = 0,$$

$$\Sigma F_n = P_1 \cos 60° - P_2 \cos 30° = \frac{30}{32.2}\,78.$$

Solution gives

$$P_1 = 62.3 \text{ lbs., and } P_2 = -48 \text{ lbs.}$$

The negative sign means that the sense of P_2 was wrongly assumed; it acts the other way (down and to the left) and is exerted by the upper end of the box.

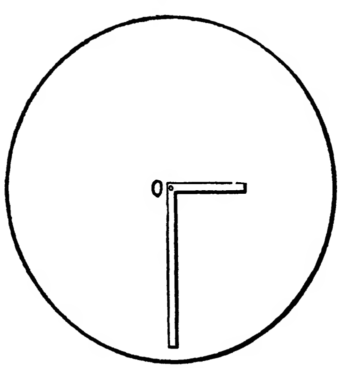


FIG. 311

EXAMPLE 5. A slender rod 3 ft. long weighing 12 lbs. is bent at right angles 1 ft. from an end and pinned at that point to the center of a horizontal table as shown in Fig. 311. The table is made to rotate uniformly about a vertical axis through its center O at the rate of 200 rev/min. It is required to determine the reaction of the pin on the rod.

Solution: The free body diagram for the rod is shown in Fig. 312. The mass-center is at C (one-third of the way from A to B) and its path is a circle with O as center and radius = 0.688 ft. The forces acting on the bar are its own weight, the upward reaction of the table, and the reaction of the pin. The first two forces (not shown) are mutually in equilibrium, since the bar has no vertical acceleration. The force exerted by the pin is represented by its components R_n and R_t, parallel respectively to the normal and the tangent to the path of C at C. Since the table has no angular acceleration, the acceleration of C consists simply of the normal acceleration $\bar{a}_n = r\omega^2 = 0.688 \times 20.94^2 = 302$ ft/sec/sec., directed as shown. Therefore

$$\Sigma F_t = R_t = 0, \quad \text{and} \quad \Sigma F_n = R_n = \frac{12}{32.2} 302, \text{ whence } R_n = 112.2 \text{ lbs.}$$

EXAMPLE 6. A body A weighing 4000 lbs. rests on a flat car that is rounding a curve of 1000 ft. radius on a track that has no elevation of the outer rail. At a particular instant the speed of the car is 40 mi/hr. and is increasing at the rate of 2 mi/hr/sec. It is required to determine the reaction of the car on the body at that instant, and to determine the least coefficient of friction between car and body if the latter is not to slip.

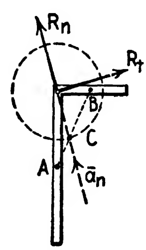


FIG. 312

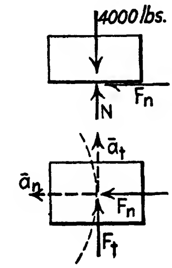


FIG. 313

Solution: The free body diagram for the body is shown in Fig. 313 (elevation above, plan below). The forces acting on the block are its weight and the reaction of the car. The latter consists of a normal pressure N (vertical) and the friction F (horizontal); F in turn consists of a tangential component F_t and a normal component F_n. The weight, N, F_t and F_n therefore make up the external system and are represented in the figure. The acceleration of the body consists of a tangential component $\bar{a}_t = 2$ mi/hr/sec. = 2.93 ft/sec/sec. and a normal component $\bar{a}_n = v^2/r$. Since $v = 40$ mi/hr. = 58.7 ft/sec., $\bar{a}_n = 58.7^2/1000 = 3.44$ ft/sec/sec. The following equations apply:

$$\Sigma F_y = -4000 + N = 0, \text{ whence } N = 4000 \text{ lbs.};$$

$$\Sigma F_n = F_n = \frac{4000}{32.2} 3.44, \text{ whence } F_n = 427 \text{ lbs.}; \quad \text{and}$$

$$\Sigma F_t = F_t = \frac{4000}{32.2} 2.93, \text{ whence } F_t = 364 \text{ lbs.}$$

The total reaction is determined by these components. Its magnitude is

$$R = \sqrt{4000^2 + 427^2 + 364^2} = 4039 \text{ lbs.}$$

The total friction is $F = \sqrt{427^2 + 364^2} = 561$ lbs. Since $N = 4000$ lbs., the coefficient of friction must be at least $561 \div 4000 = 0.14$, if there is to be no slipping.

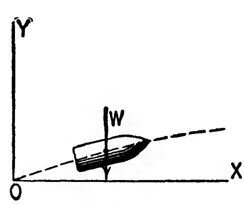


FIG. 314

EXAMPLE 7. A 14-inch rifle shoots a projectile weighing 1660 lbs. with a muzzle velocity of 2165 ft/sec. Neglecting air resistance, it is required to determine the equation of the trajectory (path of the mass-center of the projectile), the time of flight, the range, and the maximum ordinate to the trajectory when the gun is fired at an angle of elevation of 15°.

Solution: The free body diagram for the projectile is shown in Fig. 314, which also shows the x- and y-axes of reference and a portion of the trajectory. There is but one force acting on the projectile, — its own weight. Therefore

$$a_x = 0 \quad \text{and} \quad a_y = -32.2.$$

Integrating, and determining the constants of integration from the facts that when $t = 0$, $v_x = 2165 \cos 15° = 2090$ and $v_y = 2165 \sin 15° = 560$, we have

$$v_x = 2090 \text{ and } v_y = -32.2\,t + 560.$$

Integrating again and noting that when $t = 0$, $x = 0$ and $y = 0$, we have

$$x = 2090\,t \quad \text{and} \quad y = -16.1\,t^2 + 560\,t.$$

Combining these equations, we have as the equation of the trajectory

$$y = -0.00000369\,x^2 + 0.268\,x.$$

The flight ends when the projectile strikes the ground (here assumed to be a horizontal plane), that is, when $y = 0$. Setting $y = 0$, it is found that $t = 34.8$ sec., which is the time of flight. Substitution of $t = 34.8$ in the equation for x gives $x = 72{,}700$ ft., which is the range.

The projectile is at its highest point when $v_y = 0$. Setting $v_y = 0$ and solving for t gives $t = 17.4$, and substitution of this value of t in the expression for y gives $y = 4870$ ft., which is the maximum ordinate.

(Ballistic calculations such as the foregoing, in which air resistance is neglected, are of little practical value when the velocities involved are at all large. For example, the actual range of the gun described when fired with 15° elevation is but 14,950 yards, which shows that air resistance really has a great effect.)

EXAMPLE 8. Figure 315 represents diagrammatically a conical pendulum; it consists of a bob B suspended by a cord from a fixed point O, and a device, such as the pivoted arm shown, by means of which the bob and cord can be made to rotate about the vertical axis through O. When thus rotated, the bob will move outward and the cord will deflect from the vertical; then if the rotation is uniform and continuous the cord will (ordinarily) maintain some certain deflection, the bob describing a circle in a horizontal plane as indicated. It is required to determine, for any given revolutions per unit time n, and any length of cord l, the angle θ the cord deflects from the vertical. It is also required to determine the corresponding tension in the cord, pressure of the arm against the bob, and distance h from the point O to the plane in which the mass center of the bob travels.

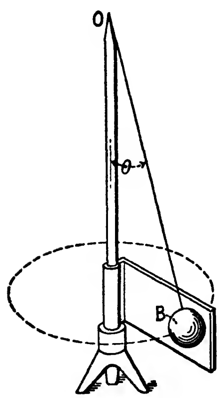


FIG. 315

Solution: The free body diagram for the bob is shown in Fig. 316. The forces acting on the bob are the pull T of the cord, the tangential pressure P of the arm, and the weight W. Since the bob is moving at constant speed in a circular path, the acceleration of its mass-center is given by $\bar{a}_n = v^2/r$. But $v = 2\,\pi r n$ and $r = l \sin\theta$, hence $\bar{a}_n = 4\,\pi^2 n^2 \sin\theta l$. The following equations may therefore be written:

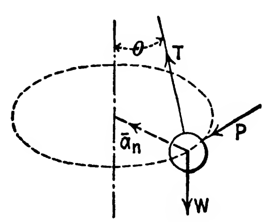


FIG. 316

$$\Sigma F_y = T\cos\theta - W = 0; \quad \Sigma F_n = T\sin\theta = \frac{W}{g}4\,\pi^2 n^2 \sin\theta l; \quad \Sigma F_t = P = 0.$$

Solution gives $$\theta = \cos^{-1}\frac{g}{4\,\pi^2 n^2 l}; \quad T = \frac{4\,\pi^2 n^2 l W}{g}; \quad P = 0.$$

It is evident from the figure that

$$h = l\cos\theta = \frac{g}{4\,\pi^2 n^2}.$$

It is important to note that for values of n less than $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$, the formula gives for $\cos\theta$ a value greater than unity. This means that for any such lower rate of rotation the cord will drop back to the vertical, no matter what l may be. (This limiting value of n is the frequency of the conical pendulum when oscillating like an ordinary pendulum; see Art. 184.) It is also of interest to note that h does not depend on l; and so if a number of bobs were suspended from the same point O by cords of different lengths, and made to rotate at the same number of revolutions per unit time, they would all travel in the same horizontal plane.

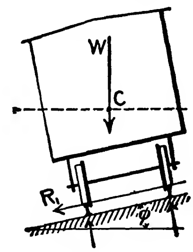


FIG. 317

EXAMPLE 9. Figure 317 represents a car rounding a curve on a railway track. It is required to determine the nature of the rail reactions on the wheels, and to ascertain the angle ϕ (Fig. 317) which, for any given speed of car v and radius of curve r, will make the flange pressure (lateral reaction of the rail against a wheel) equal to zero.

Solution: The forces acting on the car are its weight W, the reactions of the rails on the wheels, the pull P_1 of the car ahead, and the pull P_2 of the car behind. The rail reaction on each wheel may be imagined resolved into three components, — one parallel to the ties (the flange pressure), one perpendicular to the plane of the track, and one parallel to the rails. Unless the curve is very sharp the corresponding components of the reactions on all the wheels are practically parallel; they will be regarded as parallel and the resultant of the flange pressures will be called R_1, of the perpendicular components R_2, and of the other components R_3. Also, unless the curve is very sharp, P_1 and P_2 are practically parallel to the tangent to the curve under the middle of the car; they will be regarded as having this direction. The acceleration of the mass-center of the car is given by its components $\bar{a}_t$, and $\bar{a}_n = v^2/r$. The following equations apply:

$$\Sigma F_n = R_1\cos\phi + R_2\sin\phi = \frac{W}{g}\frac{v^2}{r},$$
$$\Sigma F_y = -R_1\sin\phi + R_2\cos\phi - W = 0$$
$$\Sigma F_t = P_1 - P_2 - R_3 = \frac{W}{g}\bar{a}_t.$$

Simultaneous solution of the first and second equations gives

$$R_1 = W\left(\frac{v^2}{gr}\cos\phi - \sin\phi\right) \quad \text{and} \quad R_2 = W\left(\frac{v^2}{gr}\sin\phi + \cos\phi\right).$$

Setting $R_1 = 0$, it is found that $\phi = \tan^{-1} v^2/gr$.[1]

---

[1] This formula, or some modification of it, is used to determine the proper elevation of the outer rail on railroad curves, except as noted below. The following is a practical rule deduced from the formula: "The correct superelevation for any curve is equal to the middle ordinate of a chord [of the curve] whose length in feet is 1.6 times the speed of the train in miles per hour." On the Pennsylvania Railroad the rule is modified as follows: "No speed greater than 50 miles per hour should be assumed in determining the superelevation by the above method even though higher speed may be made. No superelevation exceeding 7 inches is permissible and none exceeding 6 inches should be used except at special locations on passenger tracks." The formula was deduced on the basis that resultant flange pressure should be zero. The same formula is arrived at by making ties of the track perpendicular to the resultant pressure between the floor of the car and any object resting upon it, or perpendicular to a plumb line suspended in the car.

§ 2. Motion of a Rigid Body.

175. Translation. — In Art. 132 translation was defined as such a motion of a rigid body that each straight line of the body remains fixed in direction. It was pointed out that the motions of all points of a body in translation are alike, and that therefore by displacement, velocity or acceleration of such a body was meant the displacement, velocity or acceleration of any one of its points.

It will now be shown that *the resultant of all forces acting on a body having a motion of translation is a single force whose line of action passes through the mass-center of the body.*

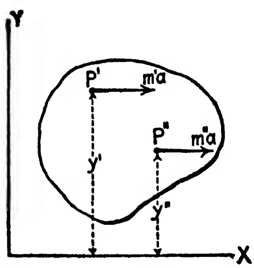


FIG. 318

Let Fig. 318 represent the body and $P'P''$, etc. its constituent particles; the external forces acting on the body are not shown. Suppose that the acceleration is directed toward the right, and let a = the magnitude of the acceleration, and m', m'', etc. = the masses of the particles respectively. Then the resultants of all the forces acting on the several particles are equal respectively to $m'a$, $m''a$, etc. These resultant forces are all directed like the acceleration, as represented in the figure, and so constitute a system of noncoplanar parallel forces. The magnitude of the resultant is found by taking the sum of the forces, thus

$$R = m'a + m''a + \cdots\cdots = Ma.$$

The line of action of the resultant (Art. 37) is determined by application of the principle of moments, thus, taking moments about the z-axis,

$$m'ay' + m''ay'' \cdots\cdots = a\,\Sigma my = M\bar{y}a,$$

where $\bar{y}$ is the y coördinate of the mass-center. The arm of R with respect to the z-axis is therefore

$$M\bar{y}a \div Ma = \bar{y}.$$

By similarly taking moments about the y-axis, it will be found that the arm of R with respect thereto is $\bar{z}$ where $\bar{z}$ is the z coördinate of the mass-center. Therefore the line of action of R passes through the mass-center of the body.

Now R is the resultant of all the forces, external and internal, acting on all the particles of the body. But the internal forces occur in pairs of equal, opposite and colinear forces and so contribute nothing to the resultant of the system. Therefore R is the resultant of the external forces that act on the body.

The torque of all the external forces about any line through the mass-center of a body having motion of translation equals zero, for the resultant of those forces (since it passes through the mass-center) has no moment about

any such line. This condition (principle) gives three independent torque equations:

$$T_x = 0, \quad T_y = 0, \quad T_z = 0,$$

where T_x, T_y and T_z denote the torques or moments of the external system for three noncoplanar lines through the mass-center. Also, the torques of the external forces with respect to *any* three lines may be equated to the torques of the resultant (Ma) about the same lines respectively.

The condition which has been established — that the torque of the external forces about any axis through the mass-center equals zero — is a criterion for motion of translation; that is, a body will not have motion of translation unless this condition is satisfied. But a body does not *necessarily* have motion of translation when this condition is satisfied, because any initial rotation that the body might have would persist. Thus when a club is hurled through the air, the only force acting on it is its own weight (air pressure being neglected), and the weight of course acts through the center of gravity; yet the stick rotates, because it was given an initial rotary motion when thrown and this rotation persists because there is nothing to stop it.

176. Typical Problems; Examples. — The common application of the principles of the preceding article is to the solution of problems in which the forces that act on a body known to have motion of translation are to be determined, or in which it is desired to ascertain whether or not, under given circumstances, a body will have motion of translation. The condition that the torque of the external forces about any line passing through the mass-center equals zero, together with the principle of the motion of the mass-center, makes possible the writing of as many as six independent equations; fewer will in general suffice for a solution.

EXAMPLE 1. Figure 319 represents a box in which a uniform bar AB of length l is placed as shown. The ends and bottom of the box are smooth. It is required to determine what acceleration to the right must be given the box in order that the bar may be prevented from slipping down.

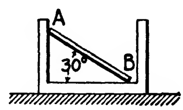


FIG. 319

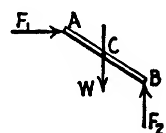


FIG. 320

Solution: The free body diagram for the bar is shown in Fig. 320. The forces acting are the weight of the bar W acting through the mass-center C, the (horizontal) pressure F_1 of the box at A, and the (vertical) pressure F_2 of the box at B. Under these forces the bar presumably moves to the right with motion of translation and with an acceleration a. According to the criterion for motion of translation the torque of the external forces about any axis through the mass-center is zero; choosing an axis perpendicular to the plane of the figure we have

$$T_C = \left(F_2 \times \frac{l}{2}\cos 30°\right) - \left(F_1 \times \frac{l}{2}\sin 30°\right) = 0.$$

The equations for the motion of the mass-center are

$$\Sigma F_x = F_1 = \frac{W}{32.2}a \quad \text{and} \quad \Sigma F_y = F_2 - W = 0.$$

Simultaneous solution of the above three equations gives a = 55.8 ft/sec/sec.

EXAMPLE 2. A straight uniform bar 12 ft. long weighing 60 lbs. is suspended in a horizontal position by two vertical ropes each 20 ft. long. One rope is attached to the bar at 4 ft. from the left end; the other is attached at the right end. The bar is raised to such a height that when released and allowed to swing freely it will have, when in its lowest position, a velocity of 16 ft/sec. It is required to determine the tension in each rope when the bar is in this position.

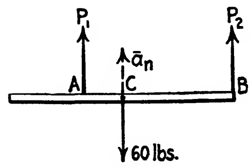


FIG. 321

Solution: The free body diagram for the bar when in its lowest position is as shown in Fig. 321. The external forces are the weight of the bar, acting through the mass-center C, and the vertical pulls P_1 and P_2 of the ropes at A and B respectively. The acceleration of the mass-center is vertical — there are no horizontal forces, therefore no horizontal acceleration — and is, simply, the normal acceleration $\bar{a}_n = 16^2/20 = 12.8$ ft/sec/sec., directed upward as shown. The following equations apply:

$$\Sigma F_y = P_1 + P_2 - 60 = \frac{60}{32.2}\,12.8, \quad \text{and}$$
$$T_C = -(P_1 \times 2) + (P_2 \times 6) = 0.$$

Solution gives $P_1 = 62.8$ lbs. and $P_2 = 20.9$ lbs.

(It should be noted that the summation of moments is not equal to zero about either A or B, the points about which moments would naturally be taken if the bar were at rest and the values of P_1 and P_2 to be found.)

EXAMPLE 3. For a certain automobile the wheel base is 122 in. and the center of gravity is 56 in. in front of the axis of the rear wheels and 35 in. above the ground. The coefficient of kinetic friction between the tires and the road surface is 0.6. It is required to determine the stopping acceleration given the car (*a*) by brakes that lock the rear wheels only and (*b*) by brakes that lock all four wheels. It is also required to determine the vertical components of the reactions on the front and rear wheels for each case.

Solution: (*a*) Except for the front wheels, the effect of which may be neglected on account of their small mass, the car has motion of translation. The free body diagram is shown in Fig. 322; the forces acting are the weight of the car W, the reactions

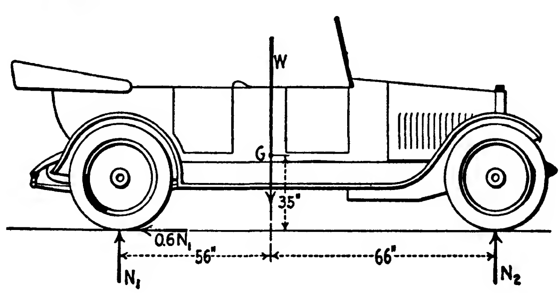


FIG. 322

of the rear wheels, regarded as a single normal component N_1 and a single friction component $0.6\,N_1$, and the reactions on the front wheels, regarded as a single vertical force N_2. The moment of the external forces about any axis through the center of gravity equals zero, there is no vertical acceleration, and so

$$T_G = N_2 \times 66 - N_1 \times 56 - 0.6\,N_1 \times 35 = 0,$$
$$\Sigma F_y = N_1 + N_2 - W = 0, \quad \text{and}$$
$$\Sigma F_x = 0.6\,N_1 = \frac{W}{32.2}\,a.$$

Solution gives $N_1 = 0.461\,W$, $N_2 = 0.539\,W$ and $a = 8.92$ ft/sec/sec.

(b) The entire car has motion of translation. Since $N_1 + N_2 = W$, the total friction is $0.6\,N_1 + 0.6\,N_2 = 0.6\,W$. The following equations apply:

$$T_G = N_2 \times 66 - N_1 \times 56 - 0.6\,W \times 35 = 0,$$
$$\Sigma F_y = N_1 + N_2 - W = 0, \quad \text{and}$$
$$\Sigma F_x = 0.6\,W = \frac{W}{32.2}a.$$

Solution gives $N_1 = 0.369\,W$, $N_2 = 0.631\,W$, and $a = 19.3$ ft/sec/sec.

It is readily shown by the methods of Art. 57 that when the car is at rest $N_1 = 0.541\,W$ and $N_2 = 0.459\,W$; therefore it is evident that braking tends to tip the car forward, increasing the load on the front wheels and decreasing the load on the rear wheels.

177. Rotation. — In Art. 134 rotation was defined as such a motion of a rigid body that one line of the body or of the extension of the body remains fixed. Angular displacement, angular velocity and angular acceleration were defined, and the relations between these quantities and the linear displacement, velocity and acceleration of any point in the rotating body developed.

It will now be shown that *the angular acceleration of a rotating body is directly proportional to the torque of the external forces about the axis of rotation, and inversely proportional to the moment of inertia of the body about the same axis.* That is, if T_0 denotes the torque of the external forces about the axis of rotation, I_0 the moment of inertia of the body with respect to that same axis, and α the angular acceleration; then α is proportional to $(T_0 \div I_0)$, and *if kinetic units* (Art. 166) *are used,* then

$$\alpha = \frac{T_0}{I_0}, \quad \text{or} \quad T_0 = I_0\alpha = Mk^2{}_0\alpha.$$

This equation we call the equation of rotation.

Let Fig. 323 represent the body and P', P'', etc. its constituent particles; the external forces acting on the body are not shown. Suppose the body to be rotating in the plane of the paper about an axis indicated by O, with a positive (counterclockwise) angular acceleration α. Let m', m'', etc. be the masses, a', a'', etc. the accelerations, and r', r'', etc. the distances from the axis of rotation of the particles P', P'', etc. respectively. Now the resultant of all forces acting on P' equals $m'a'$, and the tangential, normal and axial components of this resultant force are $m'a_t'$, $m'a_n'$, and 0 respectively. Similarly the tangential, normal, and axial components of the resultant of all forces acting on P'' are $m''a_t''$, $m''a_n''$, and 0. All the normal components are directed to ward the axis of rotation, and all the tangential components are directed counterclockwise. Now the torque of all the forces acting on P' equals the torque of $m'a_t'$ and $m'a_n'$; this torque $= m'a_t'r'$. Similarly the torque

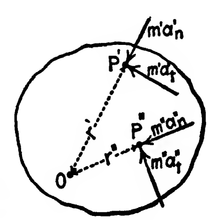


FIG. 323

of all the forces acting on $P'' = m''a_t''r''$. Hence the torque of the forces acting on all the particles equals

$$m'a_t'r' + m''a_t''r'' + \cdots\cdots = m'r'\alpha r' + m''r''\alpha r'' + \cdots\cdots = \alpha\Sigma mr^2.$$

But $\Sigma mr^2 = I_0$, the moment of inertia of the body about the axis of rotation (see Appendix B); therefore the torque of *all* the forces equals $I_0\alpha$. Now the system of forces acting on all the particles consists of *internal* and *external* forces. The *internal* forces jointly have no torque since they consist of pairs of colinear, equal, and opposite forces. Hence, the torque of the *external* forces equals $I_0\alpha$.

178. Typical Problems; Examples. — The common application of the equation of rotation is to the solution of problems in which it is required to determine the angular acceleration of a rotating body under given circumstances, or to determine the circumstances under which a body will have a certain angular acceleration. The body in question may be part of a more or less complicated system, involving several unknown quantities — velocities, accelerations, forces, dimensions, or masses. The equations of motion of the mass-center and the equation of rotation, together with the equations that express the relations between angular and linear velocity and acceleration, will in general provide a sufficient number of independent equations.

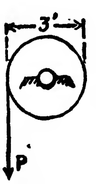


FIG. 324

EXAMPLE 1. A disc of cast iron, 4 in. thick, 3 ft. in diameter and weighing 1053 lbs., is supported on a fixed horizontal shaft 3 in. in diameter as shown in Fig. 324. A cord (weight negligible) is wrapped around the disc and a pull $P = 100$ lbs. applied thereto. It is required to determine the resulting angular acceleration of the disc.

Solution: The external forces acting on the disc are its weight, the pressure of the cord (equivalent to a tangential force $P = 100$ lbs.), and the reaction of the axle. Of these, only P has a moment about the axis of rotation. The square of the radius of gyration of the disc with respect to the axis of rotation is $\frac{1}{2}(1.5^2 + 0.125^2) = 1.133$ ft.^{2}; therefore its moment of inertia about that axis is $(1053/32.2)\ 1.133 = 37.0$ slug-ft.^{2}. Applying the equation of rotation we have

$$T_0 = 100 \times 1.5 = 37.0\,\alpha, \text{ whence } \alpha = 4 \text{ rad /sec/sec.}$$

EXAMPLE 2. Suppose that the disc of Ex. 1 is made to turn by means of a body B suspended from the cord, the weight of B being 100 lbs. It is required to determine the angular acceleration of the disc and the tension in the cord under these circumstances.

Solution: Obviously, B moves with a downward acceleration, therefore the upward pull exerted on it by the cord is less than 100 lbs. The tension in the cord being unknown, the angular acceleration of the disc cannot be found directly by means of a single equation as in Ex. 1; it is necessary to consider both bodies.

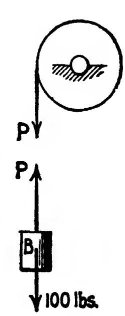


FIG. 325

The forces acting on the disc (Fig. 325, above) are its own weight, the downward pull P of the cord (equal to the tension in the cord) and the reaction of the axle. The forces acting on the body B (Fig. 325, below) are its own weight and the upward pull of the cord P.

For the disc, $\qquad T_0 = P \times 1.5 = 37\,\alpha;$

for the body B, $\Sigma F_y = 100 - P = \frac{100}{32.2} a$;

and (Art. 138) $a = 1.5\,\alpha$.

Simultaneous solution of the above three equations gives $\alpha = 3.41$ rad/sec/sec. and $P = 84.1$ lbs.

(In problems such as the foregoing it is essential that the *signs* of the forces and accelerations be consistent. Thus in the equation for the body B, the acceleration, being downward, must agree in sign with the 100 lb. force, not with P.)

EXAMPLE 3. A straight uniform bar 6 ft. long and weighing 20 lbs. is pinned at a point O, one foot from an end, to a smooth horizontal floor on which it rests. A clockwise couple of 10 ft-lbs. acting in the horizontal plane, is applied to the bar continuously for a period of 3 sec. It is required to determine the reaction of the pin on the bar at the end of that time.

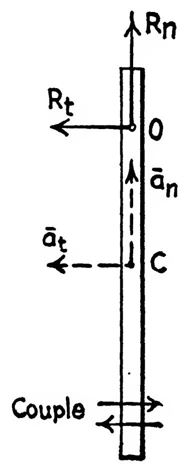


FIG. 326

Solution: The forces acting on the bar (Fig. 326) are its weight and the reaction of the floor (obviously in equilibrium and so not shown or further considered), the couple, and the reaction of the pin. Only the couple has a moment about the axis of rotation, and this moment is 10 ft-lbs., clockwise, regardless of where or how the couple is applied. The moment of inertia of the bar (Appendix B) about a vertical axis through its mass-center is $\frac{1}{12}$ (20/32.2) $6^2 = 1.86$ slug-ft.^{2}; with respect to the axis of rotation it is $1.86 + (20/32.2)\ 2^2 = 4.34$ slug-ft.^{2}. The equation of rotation is therefore

$$T_0 = 10 = 4.34\,\alpha, \text{ whence } \alpha = 2.3 \text{ rad/sec/sec.}$$

Since T_0 is constant, α is constant, and so at the end of 3 sec. $\omega = 3 \times 2.3 = 6.9$ rad/sec. Therefore the acceleration of the mass-center C consists of a tangential component $\bar{a}_t = 2 \times 2.3 = 4.6$ ft/sec/sec. and a normal component $\bar{a}_n = 2 \times 6.9^2 = 95.2$ ft/sec/sec. The equations of motion of the mass-center are therefore

$$\Sigma F_t = R_t \doteq \frac{20}{32.2} 4.6, \text{ whence } R_t = 2.86 \text{ lbs.}; \quad \text{and}$$

$$\Sigma F_n = R_n = \frac{20}{32.2} 95.2, \text{ whence } R_n = 59.2 \text{ lbs.}$$

These components, of course, completely determine the pin reaction.

EXAMPLE 4. Figure 327 represents (in plan) a straight uniform bar AB lying on the horizontal bottom of a hollow circular cylinder with vertical sides. The bar is 2 ft. long and weighs 10 lbs; the cylinder is 3 ft. in diameter; all surfaces are smooth. A horizontal force of 6 lbs. is applied to the bar at the end B, in a direction perpendicular to the axis of the bar and toward the side of the cylinder as shown. It is required to determine the angular acceleration of the bar.

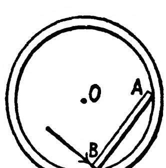


FIG. 327

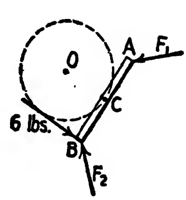


FIG. 328

Solution: Evidently the bar will slide around the cylinder, its ends in contact with the inner surface thereof; the motion is therefore a rotation about the axis of the cylinder.

The free body diagram for the bar is shown in Fig. 328. The forces acting are the weight of the bar and the reaction of the bottom of the cylinder (obviously in equilibrium and so not shown or further considered), the 6 lb. force at B, and the normal reactions F_1 and F_2 of the cylinder at A and B respectively. Of these forces, only the 6-lb. push has a moment about the axis of rotation. The moment of inertia of the bar about O (distant 1.12 ft. from its mass-center C) is

$$\frac{1}{12}\left(\frac{10}{32.2}\right)(2^2) + \left(\frac{10}{32.2}\right)(1.12^2) = 0.492 \text{ slug-ft.}^2$$

The equation of rotation is therefore

$$T_0 = 6 \times 1 = 0.492\,\alpha, \text{ whence } \alpha = 12.2 \text{ rad/sec/sec.}$$

179. Plane Motion. — In Art. 139 plane motion was defined as such a motion of a rigid body that every point of the body remains at a constant distance from a fixed plane. It was shown that with respect to angular displacement, velocity and acceleration the same definitions, expressions, units and rules of sign hold as for motion of rotation, and that plane motion may be regarded as a continuous combined translation and rotation, the translation being like the motion of any point of the body chosen as base point, and the rotation being about a line through that point.

It will now be shown that *the angular acceleration of a body in plane motion is directly proportional to the torque of the external forces about an axis through the mass-center perpendicular to the plane of the motion, and inversely proportional to the moment of inertia about that axis;* or if $\overline{T}$ denotes the torque, $\overline{I}$ the moment of inertia, and α the angular acceleration, and if kinetic units (Art. 166) are used, then

$$\alpha = \frac{\overline{T}}{\overline{I}}, \quad \text{or} \quad \overline{T} = \overline{I}\alpha.$$

Let Fig. 329 represent the body, C its mass-center, and P', P'', etc. its constituent particles; the external forces are not shown. Let m', m'', etc. be the masses of the particles respectively, r', r'' their distances from C, and a', a'', etc. their accelerations. Now if the motion of the body be regarded as a translation like the motion of C and a rotation about a base axis through C, the acceleration of P' can be regarded as consisting of three components $\overline{a}$, $r'\alpha$ and $r'\omega^2$, as indicated in the figure. Likewise the acceleration of P'' can be regarded as consisting of the three components $\bar{a}$, $r''\alpha$, and $r''\omega^2$, etc. Therefore the resultant of all the forces that act on P' consists of three components $m'\overline{a}$, $m'r'\alpha$, $m'r'\omega^2$; the resultant of all the forces that act on P'' consists of three components $m''\overline{a}$, $m''r''\alpha$, $m''r''\omega^2$, etc. The moment of all the forces that act on all the particles is the sum of the moments of all such components. It is plain that all the components $m'r'\omega^2$, $m''r''\omega^2$, etc. act through C and so have no moment about the base axis. The resultant of all components $m'a$, $m''a$, etc. passes through

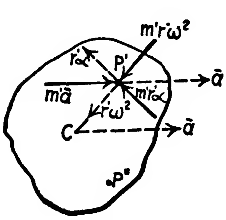


FIG. 329

the mass-center (Art. 175) and so these components have no moment about the axis. The moment of the remaining set of components is

$$m'r'\alpha r' + m''r''\alpha r'' + \cdot \cdot \cdot \cdot \cdot \cdot \cdot = \Sigma mr^2\alpha = \bar{I}\alpha.$$

Now the system of forces acting on all the particles consists of internal and external forces. The internal forces jointly have no torque, since they consist of pairs of equal, opposite and colinear forces. Hence, the torque of the *external* forces equals $\bar{I}\alpha$.

The equation $\bar{T} = \bar{I}\alpha$ contains no term depending on the motion of the mass-center; the equations of motion of the mass-center contain no term depending on the rotation of the body about that point; therefore in plane motion the rotation about the mass-center is completely independent of the motion of that point, and the angular acceleration of the body is exactly the same as though it were rotating on a fixed axis through the mass-center, perpendicular to the plane of the motion.

180. Typical Problems; Examples. — The common application of the relations established in the preceding article is to problems that involve the determining of forces which accompany a given plane motion. Thus it may be required to determine the forces acting on a rolling wheel, or on the side rod of a locomotive, or the connecting rod of a stationary engine. In general the relation $\bar{T} = \bar{I}\alpha$, together with the principle of motion of the mass-center and the geometrical relations between the motions of the parts, enables us to write out a sufficient number of equations.

EXAMPLE 1. A pair of car wheels and attached axle are pushed along a level track by a horizontal force of 150 lbs. applied at the level of the axle as shown (Fig. 330). The coefficient of friction between the wheels and the rails is sufficient to prevent slipping, and rolling resistance is negligible. The weight of the entire body is 600 lbs., the radius of gyration with respect to the axis of the axle is 9 in., and the radius of the wheels is 16.5 in. It is required to determine the linear acceleration of the wheels and axle and the least value of the coefficient of friction between wheels and rails that will prevent slipping.

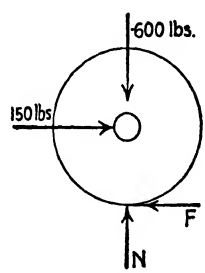


FIG. 330

Solution: The body in question (wheel-and-axle system) has plane motion, which will be regarded as a translation like that of the axle axis and a rotation about that axis. The forces acting (Fig. 330) are the weight, the push of 150 lbs., and the reactions of the rails on the wheels, regarded for convenience as a single force consisting of the vertical component N and the horizontal (friction) component F.

For rotation $\quad \bar{T} = F \times \frac{16.5}{12} = \frac{600}{32.2}\left(\frac{9}{12}\right)^2\alpha;$

for translation $\quad \Sigma F_x = 150 - F = \frac{600}{32.2}a,$

and (Art. 138) $\quad \alpha = a/(16.5/12).$

Solution of the above equations gives $a = 6.21$ ft/sec/sec. and $F = 34.5$ lbs.

Obviously $N = 600$ lbs., therefore the coefficient of friction must be at least $34.5 \div 600 = 0.06$ (about). If the coefficient of friction were less than 0.06, or if the applied force were greater than 150 lbs., the limiting friction would not be great enough to give the body an angular acceleration corresponding to its linear acceleration, and the motion would be a combination of slipping and rolling.

EXAMPLE 2. Figure 331 represents any simple vehicle, as a railroad car, being drawn along a level track by a pull P applied to the draw bar. The weight of the body of the car is W, the weight of each wheel and half-axle (the axles are attached to the wheels) is w, the radius of each wheel is r, and the radius of gyration of each set of wheels and axle is k. The force P gives the vehicle an acceleration and in general tends to tip it forward (if applied high) or backward (if applied low), in either case increasing the load on one axle and decreasing it on the other. It is required to develop a general formula for the acceleration of the vehicle and to ascertain where the draw-bar pull P should be applied in order that it may not affect the axle loads. There is no slipping, and rolling resistance may be neglected.

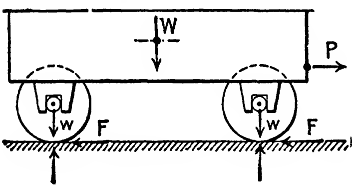


FIG. 331

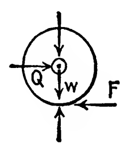


FIG. 332

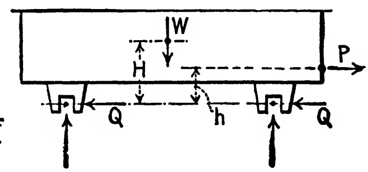


FIG. 333

Solution: The external forces exerted on the vehicle as a whole are shown in Fig. 331. They are the weight $(W + 4\,w)$, the reactions of the rails on the wheels (each represented by its vertical and horizontal components), and the draw bar pull P. The external forces acting on one wheel and half-axle are shown in Fig. 332; those acting on the body of the car are shown in Fig. 333. (Forces not lettered are not required in the following equations.) Noting that for the wheels $\alpha = a/r$, the following equations may be written:

For the rotation of each wheel $\quad \overline{T} = Fr = \dfrac{w}{g}k^2\dfrac{a}{r}$, whence $F = \dfrac{w}{g}\dfrac{k^2}{r^2}a$;

for the translation of each wheel $\quad \Sigma F_x = Q - F = \dfrac{w}{g}a$, whence $Q = \dfrac{w}{g}\left(1 + \dfrac{k^2}{r^2}\right)a$;

for the translation of the car body $\Sigma F_x = P - 4\,Q = \dfrac{W}{g}a.$

Solution gives
$$a = \frac{P}{\dfrac{W}{g} + \dfrac{4\,w}{g}\left(1 + \dfrac{k^2}{r^2}\right)}.$$

If, when P is applied, the resultant of P and the four forces Q acts through the center of gravity of the car, there will be no tendency for the car to tip either forward or backward, and therefore no change in axle loads. This condition obtains when the moment of P and the four forces Q about a horizontal transverse line through the center of gravity is zero. Let h denote the height of the draw bar pull above the plane of the axles, and H the height of the center of gravity of the car body above that plane. Then $P(H - h) = nQH$, whence

$$h = \frac{H}{1 + (4\,w/W)\,(1 + k^2/r^2)}.$$

EXAMPLE 3. A straight uniform bar AB (Fig. 334) is placed with its upper end against a smooth vertical wall and its lower end on a smooth horizontal floor, the angle of inclination to the horizontal being θ. It is required to determine the angular acceleration taken on by the bar at the instant it is released, and the pressures of the wall and floor against its ends at that same instant.

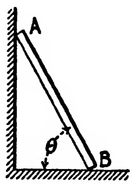


FIG. 334

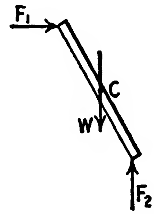


FIG. 335

Solution: Evidently the bar will slip down, its ends sliding along the wall and floor; therefore the angular acceleration is counterclockwise and the acceleration of the mass-center C is to the right and down. The free body diagram is shown in Fig. 335. The following equations apply:

$$\Sigma F_x = F_1 = \frac{W}{g}\bar{a}_x$$

$$\Sigma F_y = W - F_2 = \frac{W}{g}\bar{a}_y$$

$$\overline{T} = \left(F_2 \times \frac{l}{2}\cos\theta\right) - \left(F_1 \times \frac{l}{2}\sin\theta\right) = \frac{1}{12}\frac{W}{g}l^2\alpha.$$

Now the acceleration of C is equal to the acceleration of A plus (vectorial) the acceleration of C relative to A (Art. 155), therefore

$$\bar{a}_x = (x \text{ acceleration of } A) + \frac{l}{2}\alpha\sin\theta = \frac{l}{2}\alpha\sin\theta.$$

Similarly

$$\bar{a}_y = (y \text{ acceleration of } B) + \frac{l}{2}\alpha\cos\theta = \frac{l}{2}\alpha\cos\theta.$$

Therefore

$$F_1 = \frac{W}{g}\frac{l}{2}\alpha\sin\theta, \quad \text{and} \quad F_2 = W - \frac{W}{g}\frac{l}{2}\alpha\cos\theta.$$

Substituting these values of F_1 and F_2 in the above moment equation, and solving for α, it is found that

$$\alpha = \frac{2g\cos\theta}{l(\sin^2\theta + \cos^2\theta + \frac{1}{3})}, \quad F_1 = \frac{W\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta + \frac{1}{3}}, \quad F_2 = W - \frac{W\cos^2\theta}{\sin^2\theta + \cos^2\theta + \frac{1}{3}}.$$

181. Spherical Motion; General Motion. — The methods of this chapter are not well adapted to the discussion of spherical and general motion. These motions, in their relations to the forces that produce or modify them, are treated in Chapter XIV.

§ 3. Center of Percussion; Pendulums.

182. Center of Percussion Defined. — Figure 336 represents a body resting on a smooth horizontal surface, on which it is free to move; C is the center of gravity of the body and O any other point thereof. Suppose

a horizontal force P to be applied to the body in such a way as to be always perpendicular to OC. In general, the point O will move, but there is some point Q on OC through which the force P can be applied without causing O (or any point on a vertical axis through O) to move. This point Q is the *center of percussion* of the body with respect to the vertical axis through O.

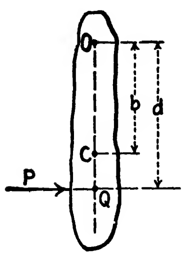


FIG. 336

Again, suppose the body to be suspended from a horizontal axle through O (Fig. 337). The force P could be applied at Q without causing (at the instant of application) a horizontal reaction of the axle on the body at O. Thus if the force be applied by means of a blow at Q (which causes a large force, but one of very short duration, ceasing before the body has moved appreciably) there will be no shock or strain upon the axle, such as would be caused by a blow struck either above or below Q.

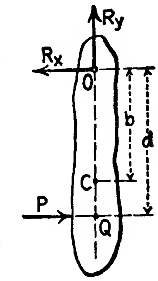


FIG. 337

Every American boy has batted a baseball a few times in such a way that the bat has "stung" his hands; such stinging is a result of impact near the hands or near the big end of the bat; in short, remote from the center of percussion of the bat (with reference to the particular axis of rotation about which the bat is being swung at the instant of impact). Such a blow results in rapid vibrations of the material of the bat, which cause the sting. Large pendulums are used in certain impact testing machines for striking a blow. To avoid the impulsive reaction at the suspension and vibrations in the pendulum, they are always so arranged that the line of action of the blow passes through the center of percussion of the pendulum.

183. Location of the Center of Percussion. — It will now be shown that the distance of the center of percussion from the axis with respect to which it is to be found is given by

$$d = \frac{b^2 + \bar{k}^2}{b} = \frac{k^2}{b}$$

where b is the distance from the axis to the center of gravity, k is the radius of gyration of the body with respect to the axis, and $\bar{k}$ is its radius of gyration with respect to a parallel axis through the center of gravity.

Consider the body as resting on the smooth horizontal plane (Fig. 336) and regard its motion as plane motion. Applying the equations $\Sigma F_x = M\bar{a}_x$ (Art. 172), and $\bar{T} = \bar{I}\alpha$ (Art. 179), and noting that point O is at rest, we have

$$\Sigma F_x = P = M\bar{a}_x = M\alpha b, \quad \text{and} \quad \bar{T} = P(d - b) = \bar{I}\alpha = M\bar{k}^2\alpha,$$

whence $$P = M\bar{k}^2\alpha/(d - b).$$

Equating the two expressions for P and solving for d gives $d = \dfrac{b^2 + \bar{k}^2}{b} = \dfrac{k^2}{b}.$

Again, consider the body suspended from an axle through O as in Fig. 337 and regard its motion as a rotation about the axis thereof. Applying the equations $\Sigma F_x = M\bar{a}_x$ and $T_0 = I_0\alpha$ (Art. 177), and noting that R_x, the horizontal component of the reaction R of the axle, is zero, we have

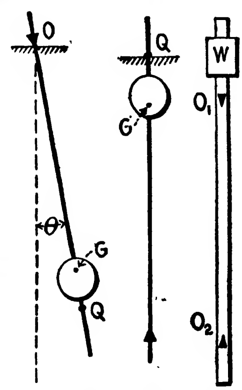
$$\Sigma F_x = P = M\bar{a} = M\alpha b, \quad \text{and} \quad T_0 = Pd = I_0\alpha = Mk^2\alpha.$$

Solution of these equations gives $d = k^2/b$.

184. Gravity Pendulum. — By this term is meant the common pendulum, that is, a body suspended on a horizontal axis so that it can be made to oscillate freely under the influence of gravity. A real pendulum is sometimes called a *compound* or *physical pendulum* to distinguish it from an imaginary one consisting of a mass-point or particle suspended by a massless cord; this latter is called a *simple* or *mathematical pendulum.* Let T = the period or time of one complete or double (to and fro) oscillation, k = the radius of gyration of the pendulum with respect to the axis of suspension, c = distance from the center of gravity of the pendulum to that axis, and 2β = the angle swept out by the pendulum in one single oscillation. Then, as will be shown presently, the period is given closely by

$$T = 2\pi\sqrt{k^2/cg}, \qquad (1)$$

provided that β is small.[1] Since β does not appear in this formula the period of any pendulum is independent of β; that is, all small oscillations of a pendulum have equal periods or, as we say, they are isochronous. When g is expressed in feet per second per second then k and c should be expressed in feet; T will be in seconds.



FIGS. 338 339 340

For the derivation of equation (1) let OG (Fig. 338) be a pendulum in any swinging position, O the center of suspension, G the center of gravity; let W = the weight of the pendulum, $c = OG$, and θ the (varying) angle which OG makes with the vertical, regarded as positive when the pendulum is on the right side of the vertical, as shown. There are three forces acting on the pendulum, — gravity, the supporting force at the knife-edge, and the pressure of the surrounding air. The moment of the first force about the axis of suspension is $Wc \sin\theta$; the moments of the other two forces we take as negligible. Hence the resultant torque on the pen-

[1] The exact value of the period is given by

$$T = 2\pi\sqrt{k^2/cg}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\beta}{2} + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 \sin^4\frac{\beta}{2} + \cdots\right].$$

If β = 8 degrees then the bracket above = 1.00122; and for smaller values of β the value of the bracket is still nearer unity. Hence the error in the approximate formula is less than one-eighth of one per cent if β does not exceed 8 degrees.

dulum in any position $= Wc \sin \theta$ practically. The angular acceleration $= d^2\theta/dt^2$ (see Art. 137); hence according to the equation of rotation

$$Wc \sin \theta = -(W/g)k^2 d^2\theta/dt^2,$$

the negative sign being introduced because $\sin \theta$ and $d^2\theta/dt^2$ are always opposite in sign. It follows readily from the preceding equation that

$$d^2\theta/dt^2 = -(cg/k^2) \sin \theta = -A \sin \theta,$$

where A is an abbreviation for cg/k^2. We will assume that the greatest value of θ, that is β, is so small that $\sin \theta$ and θ are nearly equal; then as a good approximation we may substitute θ for $\sin \theta$, and have

$$d\theta^2/dt^2 = -A\theta.$$

To integrate this simply, let $u = d\theta/dt$; then $d^2\theta/dt^2 = du/dt = (du/d\theta)(d\theta/dt) = (du/d\theta)u$, and hence

$$(du/d\theta)u = -A\theta, \quad \text{or} \quad u\,du = -A\theta\,d\theta.$$

Now integrating and replacing u by $d\theta/dt$, we get

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = -A\frac{\theta^2}{2} + C_1,$$

where C_1 is a constant of integration. Remembering that $d\theta/dt =$ the angular velocity of the pendulum, we note that where $\theta = \beta$, there $d\theta/dt = 0$; therefore for these (simultaneous) values the preceding equation becomes $0 = -\frac{1}{2} A\beta^2 + C_1$, or $C_1 = \frac{1}{2} A\beta^2$, and finally

$$d\theta/dt = \pm A^{\frac{1}{2}} \sqrt{\beta^2 - \theta^2}.$$

The positive sign is to be used when $d\theta/dt$ is positive; that is when the pendulum is swinging in the positive direction. Now let $\tau =$ the time required for the pendulum to swing out from its lowest to its highest position on the right, that is, while θ changes from 0 to β. To get a value of this time we integrate the preceding equation as follows:

$$\sqrt{A}\int_0^{\tau} dt = +\int_0^{\beta} \frac{d\theta}{\sqrt{\beta^2 - \theta^2}}, \quad \text{or} \quad \tau = \sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_0^{\beta} = \frac{\pi}{2}\sqrt{\frac{k^2}{cg}}.$$

Let $\tau' =$ the time required for a swing from the extreme right position to the lowest position, that is while θ changes from β to 0. To get this time we integrate as follows:

$$\sqrt{A}\int_0^{\tau'} dt = -\int_{\beta}^{0} \frac{d\theta}{\sqrt{\beta^2 - \theta^2}}, \quad \text{or} \quad \tau' = -\sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_{\beta}^{0} = \frac{\pi}{2}\sqrt{\frac{k^2}{cg}}.$$

Hence τ and τ' are equal, as was to be expected. Finally, the time of one complete oscillation $= 4\,\tau = 2\,\pi\,\sqrt{k^2/cg}$, as was to be shown.

Period of a Simple Pendulum. — Let $l =$ the length of the pendulum; that is, the distance from its point of suspension to the mass-point (or very small bob). Then $c = l$ and $k = l$; hence according to (1)

$$T = 2\,\pi\,\sqrt{l/g} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Equivalent Simple Pendulum. — A simple pendulum whose period is the same as that of any particular or given physical pendulum is said to be equivalent to the physical pendulum. The length of such simple pendulum is called the length of the physical pendulum.

As before let k and c refer to any physical pendulum; also let l' be the length of the equivalent simple pendulum. Equating the formulas for periods and simplifying gives

$$l' = k^2/c. \qquad (3)$$

Center of Oscillation. — That point of an actual pendulum into which its entire mass might (in imagination) be concentrated without changing the period is called the center of oscillation of the pendulum. If k and c refer as before to the real pendulum, and Q (Fig. 338) is its center of oscillation

$$OQ = l' = k^2/c. \qquad (4)$$

Also if $\bar{k}$ denote the radius of gyration of the pendulum with respect to a line through its center of gravity and parallel to its axis of rotation, then $k^2 = \bar{k}^2 + c^2$, and

$$GQ = \bar{k}^2/c. \qquad (5)$$

"The centers of suspension and oscillation of a pendulum are interchangeable," that is if the pendulum shown in Fig. 338 be turned end for end and then suspended from Q as shown in Fig. 339, then in the new position O is the center of oscillation. Proof: Let Q' (not shown) be the center of oscillation, corresponding to Q (Fig. 338). Then according to (4),

$$GQ' = \frac{\bar{k}^2}{QG} = \frac{\bar{k}^2}{\bar{k}^2/c} = c;$$

hence Q' coincides with O. It follows from the property of interchangeability that the periods of a pendulum when suspended from O and Q are equal.

Determination of g. — Pendulums are used to determine local values of g. To make a determination one has only to determine the period T of a pendulum, and its length k^2/c, and then compute g from formula (1).

The length k^2/c cannot be determined accurately for an ordinary pendulum, so a special form (shown in principle in Fig. 340) has been designed for the purpose. O_1 and O_2 are two knife-edges as shown at a known distance apart; W is a weight which can be slid along the stem and clamped where desired. By repeated trials, shifting the weight as necessary, the periods of oscillation for O_1 and O_2 suspension are made equal; then the length k^2/c is equal to O_1O_2.

By means of the pendulum described, the value of g for Washington was determined to be 980.100 centimeters per second per second. Values of g at many other places have been determined more simply — by comparing the periods of a more ordinary pendulum at Washington and the other

places. This comparison is based on the principle that the squares of the periods of any pendulum at two different places are inversely proportional to the values of g at those places; hence if T_w and T = the periods at Washington and some other station and g = the acceleration at the latter place, then $g = 980.1\ (T_w/T)^2$.

185. Torsion Pendulum. — This consists of a heavy bob suspended vertically by means of a light elastic wire, the wire being firmly embedded in the bob and in its support. Any horizontal couple applied to the bob will turn the bob and twist the wire. If the couple is not too large — so as not to stress the wire beyond its "elastic limit" — then the angular displacement of the bob will be proportional to the moment of the couple applied. That is, if C and C' = the moments of two couples applied successively and θ and θ' are the corresponding angular displacements produced by the couples, then $\theta/\theta' = C/C'$. Hence, the moment C required to produce any displacement is given by $C = (C'/\theta')\theta$. In any displaced position of the bob, the wire exerts a couple on the bob equal to the applied couple.

If the bob were released from any position of (moderate) angular displacement β, it would oscillate under the influence of the couple exerted by the wire. We will assume that this (varying) couple follows the law expressed above. Then the equation of motion (rotation) for the bob would be (see Art. 177) $C = I\alpha$, where I = moment of inertia of the bob with respect to the axis of the wire and α = the (varying) angular acceleration. Since $\alpha = d^2\theta/dt^2$, and θ and $d^2\theta/dt^2$ are opposite in sign, the equation can be written

$$\frac{C'}{\theta'}\theta = -I\frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -B\theta,$$

where B is an abbreviation for $(C'/\theta') \div I$. This last equation is just like the equation $d^2\theta/dt^2 = -A\theta$ of Art. 184, relating to the motion of a gravity pendulum, except that B appears in one equation and A in the other. Hence the formulas in Art. 184 apply to this section if A be changed to B. Thus the time of one quarter complete oscillation of a torsion pendulum is

$$\tau = \sqrt{\frac{1}{B}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_0^\beta = \frac{\pi}{2}\sqrt{\frac{I}{C'/\theta'}}.$$

The period (one complete to and fro oscillation) equals 4τ, or

$$T = 2\pi\sqrt{I \div (C'/\theta')}.$$

$I = Mk^2 = (W/g)\,k^2$ where W = weight and k = radius of gyration of the bob with respect to the axis of the wire. If, in a numerical case, W is taken in pounds and g in feet per second per second, k should be expressed in feet, C' in foot-pounds, and θ' (always) in radians; T will be in seconds. C'/θ' (the ratio of the moment of any twisting couple to the angle of twist

produced) is a measure of the torsional stiffness of the wire, for that ratio is the moment required for twisting per radian of twist. The above formula for T shows that the stiffer the wire the less the period.

§ 4. D'Alembert's Principle

186. Statement of the Principle. — The resultant of all the forces acting on any particle of a body is called the *effective force* for that particle. Its magnitude equals the product of the mass and acceleration of the particle; its direction is the same as that of the acceleration. The group of effective forces for all the particles of a body is called the *effective system* (of forces) for the body. It should be noted that these forces are fictitious or imaginary, equivalent respectively to the actual forces acting upon the particles.

Now the actual forces acting on all the particles include all the internal and external forces acting on the body; the internal forces occur in pairs of equal, opposite and colinear forces and so mutually balance; hence the actual forces acting on all the particles are equivalent to the external forces acting on the body. And so: *The external system of forces and the effective system of forces for any body are equivalent; therefore the external system and the reversed effective system are in equilibrium.*

D'Alembert's principle is not a new and independent principle of Dynamics. It is the basis of a new or independent *plan* of analysis of dynamic problems which may be described as follows: (i) Ascertain the resultant, or simple equivalent, of the effective system for the body; (ii) regard the body as in equilibrium under the combined action of this resultant effective system reversed and the actual external forces; and (iii) solve the combined system for the unknown quantities desired by the appropriate equilibrium equations.

The problem is, so to speak, transformed from a problem in Dynamics to a problem in Statics. This transformation constitutes the particular novel feature of the D'Alembert plan. The difficulty, if any, in a particular problem is to find the resultant of the effective system. That found, the rest is easy. Some prefer to use the plan quite generally in Dynamics. It is especially useful in determining stresses in moving bodies, because the subject of stresses ("Strength of Materials") is developed for the most part with reference to bodies at rest. Thus D'Alembert's plan brings the problems of dynamic stress within the scope of "Strength of Materials."

187. Resultant of Effective System. — Certain information pertaining to this subject is developed in some preceding articles, but the term "effective force" was not specifically used there. We now recall certain results already reached and extend the discussion somewhat.

Let P' (Fig. 341) be any particle of a body not shown, m' its mass, a' its acceleration, a_x', a_y' and a_z' components of the acceleration parallel to any fixed coördinate axes OX, OY and OZ. The x, y and z components of the effective force ($m'a'$) for P' are as indicated in the figure. Then the x

component of the effective system is $m'a_x' + m''a_x'' + \cdot \cdot \cdot \cdot \cdot \cdot \cdot = M\bar{a}_x$ (proved in Art. 172) and the x, y and z components of the effective system for the body are respectively

$$M\bar{a}_x,\ M\bar{a}_y \text{ and } M\bar{a}_z.$$

We have already found values of the moments of the effective system about certain axes for three cases: (i) For translation, the moment about any line through the mass-center is zero (Art. 175). (ii) For rotation about a fixed axis, the moment about the axis of rotation is $I\alpha$ (Art. 177). (iii) For any plane motion, the moment about a line perpendicular to the plane of the motion and through the mass-center is $\bar{I}\alpha$ (Art. 179).

Only in one case, translation, have we *fully* determined the resultant of the effective system. It equals Ma, is directed like a, and acts through the mass-center. In following articles we complete the determination of the resultant of the effective system for any body having rotation, and for certain bodies having plane motion.

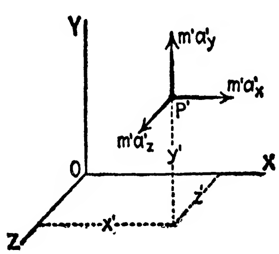


FIG. 341

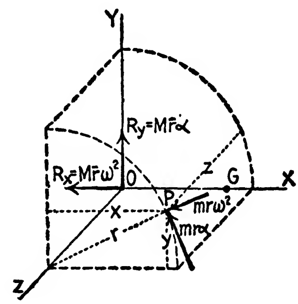


FIG. 342

188. Rotation. — In this article we reduce the effective system to a simpler or the simplest (resultant) form.

Let OZ (Fig. 342) be the axis of rotation of a body, not shown, and G its mass-center. GO is taken as axis of x. Then the coördinate frame $OXYZ$ rotates with the body. According to Art. 41 any system of forces can be reduced to a single force acting through any chosen point and a single couple. Let R and C denote this force and couple, and O the chosen point; also let R_x, R_y and R_z denote the components of R along the coördinate axes, and C_x, C_y and C_z the components of C perpendicular to the axes respectively. With coördinates chosen as described, $\bar{a}_x = -r\omega^2$, $\bar{a}_y = \bar{r}\alpha$ and $\bar{a}_z = 0$ (see Art. 138); hence (see Art. 172)

$$R_x = -M\bar{r}\omega^2, \quad R_y = M\bar{r}\alpha, \quad \text{and} \quad R_z = 0 \quad . \ . \ (1),\ (2),\ (3)$$

as indicated in Fig. 342. Also (according to Art. 177)

$$C_z = I_z\alpha, \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (4)$$

where I_z denotes moment of inertia about the z-axis.

Values of C_x and C_y must be determined anew. For this purpose let P (Fig. 342) represent any particle of the body, and let m denote the mass of this particle, r its distance from the axis of rotation, and x, y and z its coördinates with respect to the assumed coördinate axes. As shown in Art. 177, the resultant of all forces acting on P — that is, the effective force on P — consists of a tangential component $mr\alpha$ and a normal component $mr\omega^2$. The effective force for every particle can be similarly represented by its tangential and normal components.

The moment about OX of the effective force on the particle P (found by taking the moments of the x and y components of $mr\alpha$ and of $mr\omega^2$) is

$$-mr\alpha\cos\theta\, z + mr\omega^2 \sin\theta\, z = -mxz\alpha + myz\omega^2.$$

The moment about OX of the effective forces on all the particles is therefore

$$C_x = \Sigma(-mxz\alpha + myz\omega^2) = -\alpha K_{xz} + \omega^2 K_{yz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where K_{xz} denotes the product of inertia of the body with respect to the coördinate planes from which x and z are measured; similarly K_{yz} (see Appendix B). The moment about OY of the effective force on P is similarly found to be

$$-mr\alpha\sin\theta\, z - mr\omega^2\cos\theta\, z = -myz\alpha - mxz\omega^2$$

and the moment of the effective system to be

$$C_y = \Sigma(-myz\alpha - mxz\omega^2) = -\alpha K_{yz} - \omega^2 K_{xz} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Three special cases may be distinguished for which the above equations become greatly simplified.

(1) *The body is homogeneous, has a plane of symmetry, and rotates about an axis perpendicular to that plane.* In this case K_{xz} and K_{yz} both vanish (Appendix B) and so $C_x = 0$ and $C_y = 0$. The resultant of the effective system thus reduces to the force $R = M\bar{a}$ and the couple $C_z = I_z\alpha$, and these can be compounded into a single force $R = M\bar{a}$, acting in the xy plane and having a moment about O equal to $I_z\alpha$ (Art. 32).

If the body rotates *uniformly* ($\alpha = 0$), $C_z = 0$, and the resultant of the effective system reduces to the force $R = M\bar{a}$. But when $\alpha = 0$, $\bar{a} = \bar{r}\omega^2$, and so $R = M\bar{r}\omega^2$ and acts through O and through the mass-center.

When the mass-center is in the axis of rotation ($\bar{r} = 0$), $\bar{a} = 0$, $R = 0$, and the resultant reduces to the couple $C_z = I\alpha$.

When both α and $\bar{r} = 0$, the resultant is nil and the effective system vanishes.

(2) *The body is homogeneous, has a line of symmetry, and rotates about an axis parallel to that line.* In this case, as in (1), K_{xz} and K_{yz} vanish, and so $C_x = 0$ and $C_y = 0$. All the remarks made under case (1) about the resultant of the effective system hold also for this case.

(3) *The body is homogeneous, has a plane of symmetry, and rotates about an axis in that plane.* In this case K_{yz} vanishes, but K_{xz} does not generally.

Hence $C_x = -K_{xz}\alpha$ and $C_y = -K_{xz}\omega^2$. The resultant of the effective system *in general* reduces to a single force R which pierces the plane of symmetry at a distance $K_{zx}/M\bar{r}$ from the x-axis and a distance $I_z/M\bar{r}$ from the z-axis. Proof: Let MN, Fig. 343, represent the plane of symmetry, coincident with the xz plane. The effective system is represented by the couples C_x, C_y and C_z (indicated by the curved arrows) and by the components $M\bar{r}\alpha$ and $M\bar{r}\omega^2$ of R acting at O. C_x and $M\bar{r}\alpha$ can be compounded into a force $M\bar{r}\alpha$ acting in the zy plane at a distance $K_{xz}/M\bar{r}$ from the x-axis (in order that it may have a moment about that axis equal to C_x). Similarly, the couple C_y and the force $M\bar{r}\omega^2$ can be compounded into a parallel force $M\bar{r}\omega^2$ acting in the xz plane at a distance $K_{xz}/M\bar{r}$ from the y-axis. These forces are seen to be equidistant from the xy plane and hence concurrent in the z-axis. The force $M\bar{r}\alpha$ and the couple C_z can be compounded into a parallel force $M\bar{r}\alpha$ concurrent with $M\bar{r}\omega^2$ at Q, distant $I_z/M\bar{r}$ from the z-axis, and finally these two forces can be compounded at Q into a single force, the resultant of the effective system.

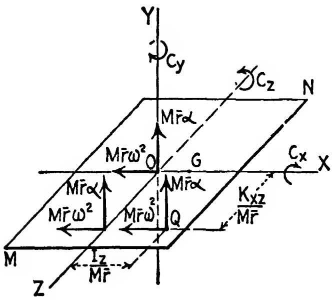


FIG. 343

If the body rotates uniformly ($\alpha = 0$) then the resultant consists of the single component $M\bar{r}\omega^2$, acting in the plane of symmetry, perpendicular to the axis of rotation, and at a distance $K_{xz}/M\bar{r}$ from the xy plane.

If the mass-center lies in the axis of rotation ($\bar{r} = 0$), the resultant does not reduce to a single force, but $R = 0$, $T_x = 0$, and the resultant of the effective system is the couple $C_y = -K_{xz}\omega^2$.

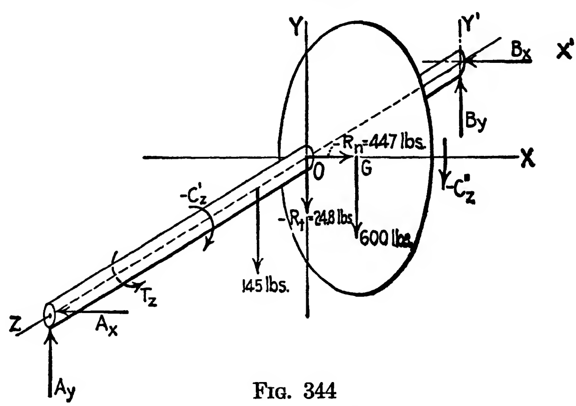


FIG. 344

EXAMPLE 1. A horizontal steel shaft 6 ft. long and 3 in. in diameter is supported in bearings at each end and has a heavy disc eccentrically mounted upon it 2 ft. from one end (Fig. 344). The center of the disc is 8 in. from the center of the shaft; the shaft

weighs 145 lbs.; the disc weighs 600 lbs., and its mass-center may be taken to coincide with its geometrical center. By means of a couple T_z applied to the shaft the whole system is given an angular acceleration of 2 rad/sec/sec. At the instant the angular velocity becomes 6 rad/sec., the center of the disc is level with the axis of the shaft as shown, and is moving upward. It is required to determine the reactions of the bearings on the shaft at that instant.

Solution: A coördinate frame $OXYZ$ is assumed; OZ is the axis of rotation and OX contains the mass-center G of the disc. We now ascertain the effective system of forces, and in doing this it is convenient to consider the shaft and the disc separately. The shaft rotates about a line of symmetry, and for it the effective system reduces to a couple C_z' (that part of the applied couple T_z which goes to accelerate the shaft). The disc rotates about an axis parallel to a line of symmetry; hence for it the effective system reduces to a couple C_z'' (that part of T_z which goes to accelerate the disc) and a force $R = M\bar{a}$, which acts through O and consists of the components $R_t = M\bar{a}_t$ and $R_n = M\bar{a}_n$. Since $\alpha = 2$ rad/sec/sec., $\bar{a}_t = \frac{8}{12} \times 2 = 1\frac{1}{3}$ ft/sec/sec.; since $\omega = 6$ rad/sec., $\bar{a}_n = \frac{8}{12} \times 6^2 = 24$ ft/sec/sec. Therefore

$$R_t = \frac{600}{32.2}\, 1\tfrac{1}{3} = 24.8 \text{ lbs.,} \quad \text{and} \quad R_n = \frac{600}{32.2}\, 24 = 447 \text{ lbs.}$$

The effective system *reversed* ($-C_z'$, $-C_z''$, $-R_t$ and $-R_n$) is represented on the figure; the external system of forces (weight of shaft, weight of disc, applied couple T_z, and components of the bearing reactions A_x, A_y, B_x and B_y) are also represented. The reversed effective system and the external system together are in equilibrium; they constitute a set of nonconcurrent, noncoplanar, nonparallel forces. The methods of Art. 59 apply, and solution is effected as follows:

$\Sigma M_y' = (447 \times 2) - (A_x \times 6) = 0$, whence $A_x = 149$ lbs.
$\Sigma F_x = 447 - A_x - B_x = 0$, whence $B_x = 298$ lbs.
$\Sigma M_x' = (600 \times 2) + (145 \times 3) + (24.8 \times 2) - (A_y \times 6) = 0$, whence $A_y = 281$ lbs.
$\Sigma F_y = -24.8 - 600 - 145 + A_y + B_y = 0$, whence $B_y = 489$ lbs.

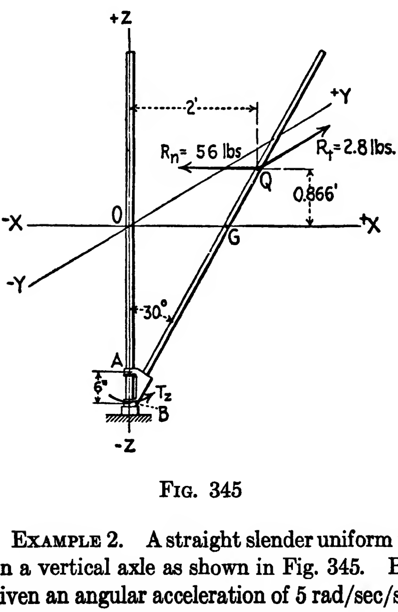


FIG. 345

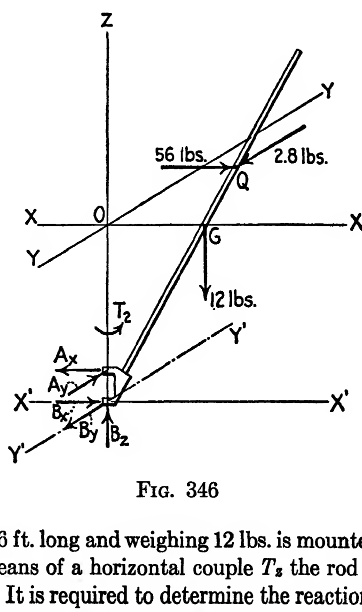


FIG. 346

EXAMPLE 2. A straight slender uniform rod, 6 ft. long and weighing 12 lbs. is mounted on a vertical axle as shown in Fig. 345. By means of a horizontal couple T_z the rod is given an angular acceleration of 5 rad/sec/sec. It is required to determine the reactions of the shaft on the rod at the instant when the angular velocity becomes 10 rad/sec.

(The mass of the attachment by means of which the rod is mounted may be neglected.)

Solution: A coördinate frame $OXYZ$ is assumed, OZ being the axis of rotation and OX containing the mass-center G of the rod. Evidently the xz plane is a plane of symmetry, and so the problem comes under case (3) above — rotation about an axis in a plane of symmetry. The effective system therefore reduces to a single force R with components $R_t = M\bar{a}_t$ and $R_n = M\bar{a}_n$, which pierces the plane of symmetry at a point Q distant $K_{xz}/M\bar{r}$ from the x-axis and distant $I_z/M\bar{r}$ from the z-axis. To determine the position of Q, I_z and K_{xz} must be found. The former is given by

$$I_z = \tfrac{1}{12} Ml^2 \sin^2 30° + (M \times 1.5^2) = 1.12 \text{ slug-ft.}^2$$

(Appendix B). To determine K_{xz}, let dm be the mass of an elementary length dL of the rod. Since the rod weighs 2 lb/ft. of length, $dm = (2/32.2)\,dL = 0.0621\ (dz/\cos 30°) = 0.0717\ dz$ (slugs). Also, $x = 1.5 + z \tan 30° = 1.5 + 0.577\,z$. Therefore

$$K_{xz} = \Sigma xzdm = 0.0717 \int_{-2.6}^{+2.6} (1.5 + 0.577\,z)\,zdz = 0.484 \text{ slug-ft.}^2$$

Therefore the distances of Q from the z-axis and the x-axis are

$$1.12 \div \left(\frac{12}{32.2} \times 1.5\right) = 2 \text{ ft.,} \quad \text{and} \quad 0.484 \div \left(\frac{12}{32.2} \times 1.5\right) = 0.866 \text{ ft.}$$

Q must lie *above* the x-axis and to the *right* of the z-axis, in order that the moment of R_t and R_n about the x-axis shall be negative (equal to $C_x = -K_{xz}\alpha$) and their moment about the z-axis positive (equal to $C_z = I_z\alpha$).

Since $\alpha = 5$ rad/sec/sec., $\bar{a}_t = 1.5 \times 5 = 7.5$ ft/sec/sec. Since $\omega = 10$ rad/sec., $\bar{a}_n = 1.5 \times 10^2 = 150$ ft/sec/sec. Therefore

$$R_t = \frac{12}{32.2}7.5 = 2.8 \text{ lbs.} \quad \text{and} \quad R_n = \frac{12}{32.2}150 = 56 \text{ lbs.}$$

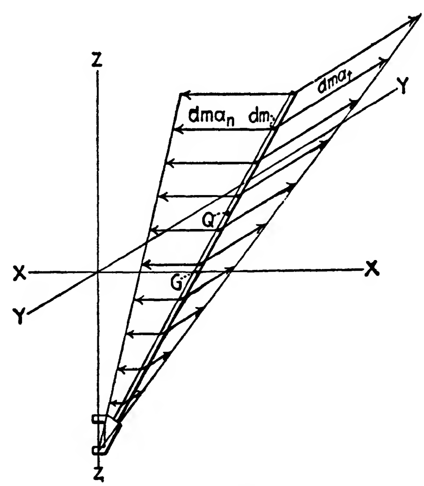


FIG. 347

The effective system is, then, completely represented by the components R_t and R_n acting at the point Q, as shown in Fig. 345. In Fig. 346 are represented the *reversed* effective system and the external forces acting on the rod (couple T_z, weight of rod, and components of the axle reactions A_x and A_y at A, B_x, B_y and B_z at B). The reversed effective system and the external system together are in equilibrium. They constitute

a set of noncoplanar, nonconcurrent nonparallel forces. The methods of Art. 59 apply and solution is effected as follows:

$$\Sigma M_{X'} = (2.8 \times 3.46) - (A_y \times 0.5) = 0, \text{ whence } A_y = 19.4 \text{ lbs.}$$
$$\Sigma F_y = 19.4 - 2.8 - B_y = 0, \text{ whence } B_y = 16.6 \text{ lbs.}$$
$$\Sigma M_{Y'} = (56 \times 3.46) + (12 \times 1.5) - (A_x \times 0.5) = 0, \text{ whence } A_x = 424 \text{ lbs.}$$
$$\Sigma F_x = 56 - 424 + B_x = 0, \text{ whence } B_x = 368 \text{ lbs.}$$

(The above problem can also be solved directly from a consideration of the effective forces on the constituent particles of the rod. For each elementary portion of the rod of length dL and mass dm, the components of the effective force are dma_t and dma_n, as represented in Fig. 347. Since $a_t = x\alpha$, and $a_n = x\omega^2$, it is evident that both components increase uniformly as the particle is taken nearer the outer end of the rod. The effective forces — or rather components — therefore vary linearly, as shown, and the resultant of each set therefore acts through a point one third of the way down from the upper end of the bar. Q may readily be located in this way. Since it is known that $R_t = M\bar{a}_t$ and $R_n = M\bar{a}_n$, the effective system is thus determined.

This method is not always practicable, but in any case it is well to visualize the effective forces on the constituent particles, or to represent them somewhat as in Fig. 347, since this gives a general idea as to the forces involved, the sense of couples, etc.

189. Plane Motion. — In this article, we reduce the effective system to a simple resultant form, considering only the special case of a homogeneous body symmetrical with respect to the plane of the motion.

In such a case, for any particle P on one side of the plane of symmetry, there is one at an equal distance directly opposite on the other side. Since the accelerations of these two particles are exactly the same, the effective forces for the particles are equal and have like directions; the resultant of the effective forces for the pair of particles (and for all such pairs) therefore lies in the plane of symmetry, and the effective system may be regarded as coplanar. Hence the effective system can be reduced to a force R acting through any chosen point and a couple C, both *in* the plane of symmetry.

We choose the mass-center G as the point. Then as proved presently: $R = M\bar{a}$ and acts in the direction of $\bar{a}$, and $C = \bar{I}\alpha$ (Fig. 348), where $\bar{a}$ denotes acceleration of the mass-center, α the angular acceleration of the body, and $\bar{I}$ the moment of inertia of the body about a line through the mass-center and perpendicular to the plane of the motion. Proof: According to Art. 172

$$R_x = M\bar{a}_x, \quad R_y = M\bar{a}_y, \quad R_z = M\bar{a}_z = 0.$$

Hence $R = M\sqrt{\bar{a}_x^2 + \bar{a}_y^2} = M\bar{a}$. The angle R makes with the x-axis is $\tan^{-1}(\bar{a}_y \div \bar{a}_x)$ which is also the angle $\bar{a}$ makes with the x-axis; and so R is directed like $\bar{a}$. In Art. 179 it is shown that the moment of the effective system about the z-axis (see Fig. 348) is $\bar{I}\alpha$; hence the moment of R and C together is $\bar{I}\alpha$; but the moment of R about that axis is zero and so $C = \bar{I}\alpha$.

R and C can be combined into a single force (Fig. 349) acting parallel to $\bar{a}$ at such side of and distance from the mass-center that its moment about the mass-center equals $\bar{I}\alpha$.

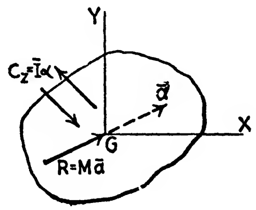


FIG. 348

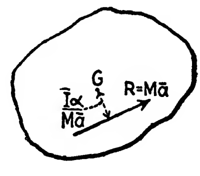


FIG. 349

EXAMPLE. Suppose that the connecting rod whose motion was discussed in the example of Art. 142 is part of the mechanism shown in Fig. 350. The weight of the reciprocating parts (piston, piston rod and crosshead) is 280 lbs. The weight of the connecting rod is 360 lbs.; its mass-center G is 2 ft. from B and its moment of inertia about an axis through G perpendicular to the plane of motion is 40 slug-ft.^{2}. The total net steam pressure P is 8000 lbs. at the instant that the rod has the position, velocity and acceleration it had in the example referred to above. It is required to determine the pin reactions at B and C at that instant.

Solution: In the example referred to it was found that the angular velocity and acceleration of the rod were respectively $\omega = 2.62$ rad/sec. (counterclockwise) and

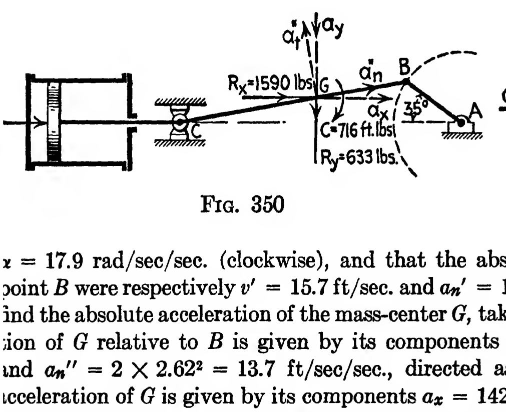


FIG. 350

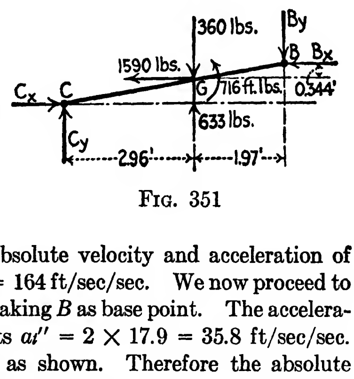


FIG. 351

$\alpha = 17.9$ rad/sec/sec. (clockwise), and that the absolute velocity and acceleration of point B were respectively $v' = 15.7$ ft/sec. and $a_n' = 164$ ft/sec/sec. We now proceed to find the absolute acceleration of the mass-center G, taking B as base point. The acceleration of G relative to B is given by its components $a_t'' = 2 \times 17.9 = 35.8$ ft/sec/sec. and $a_n'' = 2 \times 2.62^2 = 13.7$ ft/sec/sec., directed as shown. Therefore the absolute acceleration of G is given by its components $a_x = 142$ ft/sec/sec. and $a_y = -56.6$ ft/sec/sec. (found simply by summing up a_n', a_t'' and a_n'' along the horizontal and vertical).

The effective system of forces on the rod therefore reduces to a clockwise couple $C = \bar{I}\alpha = 40 \times 17.9 = 716$ ft-lbs., and a force consisting of the components $R_x = \frac{360}{32.2} 142 = 1590$ lbs. and $R_y = \frac{360}{32.2} 56.6 = 633$ lbs. This effective system is represented on Fig. 350. The *reversed* effective system and the external system (weight of rod, and components B_x, B_y, C_x and C_y of the pin reactions) are represented on the free body diagram of the rod shown in Fig. 351. The combined systems are in equilibrium, but present too many unknowns to permit solution. Considering the reciprocating mass, however, which has an acceleration to the right of 153 ft/sec/sec., we may write $\Sigma F_x = 8000 - C_x = \frac{280}{32.2} 153$, whence $C_x = 6670$ lbs. Solution may now readily be completed,

thus:

$$\Sigma F_x = 6670 - 1590 - B_x = 0, \text{ whence } B_x = 5080 \text{ lbs.}$$
$$\Sigma M_B = 716 - (633 \times 1.97) + (360 \times 1.97) - (1590 \times 0.344) + (6670 \times 0.86) - (C_y \times 4.93) = 0, \text{ whence } C_y = 1090 \text{ lb.}$$
$$\Sigma F_y = 1090 - 633 - 360 - B_y = 0, \text{ whence } B_y = 97 \text{ lb.}$$

190. "Running Balance." — Any one who has observed machines in operation has probably encountered some that shake or vibrate more or less. Almost any automobile engine idling in a car at rest affords a good example because the car body, which supports the engine, can vibrate quite freely. Such vibrations are apparently due to the motion of the moving parts of the machine; indeed, every moving part of a machine, in general, develops *some* "disturbance." Rotating parts can be made so as to produce little or no disturbance, and in some cases other parts of a machine can be so fashioned and arranged that their disturbances neutralize each other more or less. We now explain more precisely what we mean by disturbances in the case of rotating bodies, and then develop the theory for reducing them to zero or putting the body into "running" or "dynamic balance."

Consider the rotation of the simple body represented in Fig. 352, consisting of a homogeneous disk keyed at mid-length to a shaft AB supported in frictionless bearings, A and B, not shown. G is the mass-center of the disk and shaft. Imagine this body with a suitable base on which the bearings rest, transported into remote space where it would be practically free from gravity, and then put into rotation and left to itself. It would continue to rotate at constant speed; the only forces acting on it would be the reactions of the bearings on the shaft, always parallel but opposite to the reversed effective force. The pressures of the shaft on the bearings would always be equal and opposite to the reactions on the shaft, and since these pressures would continually change direction they would shake the base. Since there are no other external forces acting on the rotating body, the reactions on the shaft *and* the pressures on the bearings are said to be "due to the motion"; the latter are called also "inertia forces." In general, *any body* whatsoever thus rotating exerts pressures, due to the motion, on its bearings. And it should be noted that these pressures are precisely equivalent to the reversed resultant effective force for the body. Hence, if a rotating body is so fashioned or mounted that its reversed resultant effective force is nil, there are no pressures on the bearings and hence no shaking of the frame; the body is then said to be in "running" or "dynamic balance."

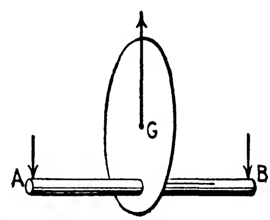


FIG. 352

Imagine now any body on a shaft in bearings as above at the Earth and rotating at constant speed under the action of any external forces, including

gravity and the reactions of the bearings. The reactions may each be regarded as consisting of two components, one due to the other external forces and one due to the motion. The first components may or may not change in magnitude and in direction, depending on circumstances; the second do change direction just as in remote space. And so if the body were in dynamic balance there would be no component bearing pressure due to the motion, and to this extent at least the shaking would be reduced.

It is possible to put any (rotating) body into running balance by the addition (or subtraction) of material at one point in each of two arbitrarily chosen planes perpendicular to the axis of rotation. Proof: Let AB (Fig. 353)

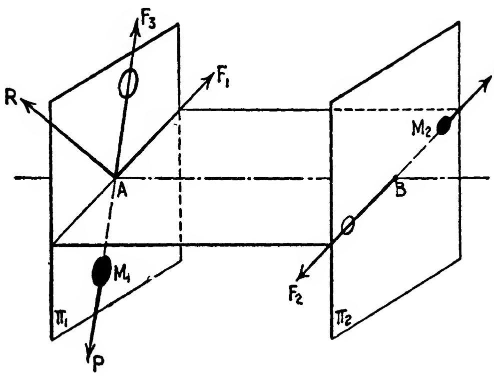


FIG. 353

be the shaft supporting the body, not shown, and π_1 and π_2 the chosen planes. The reversed effective force for any particle of the body is a single force directed outward from the center of rotation of that particle. The reversed effective system can be reduced to a single force R acting at A, say, and a single couple C (see Art. 188). Since the reversed effective forces have no components along the axis (of the shaft), R has no component along the axis and so it must lie in the plane π_1, say, as represented. Since the reversed effective forces have no moment about the axis, C has no moment about the axis; therefore its plane includes the axis, and F_1 and F_2 (equal, parallel, and opposite) may represent C. Now R and F_1 may be compounded into a single force F_3 acting through A and in the plane π_1; hence F_2 and F_3 are equivalent to the reversed effective system. Clearly masses M_1 and M_2 could be added to the body somewhere on the lines of action of F_3 and F_2 about as shown, at such distances from A and B that the reversed effective forces, P and Q, of M_1 and M_2 would balance F_3 and F_2 respectively. Hence the reversed effective forces for the original body with the added material would be in equilibrium, or their resultant would be zero; the altered body would be in running balance.

If the planes π_1 and π_2 are chosen so that material could be removed from the body at these planes, then the reversed effective system could be put

into equilibrium by removing masses M_1 and M_2, but on the other sides of A and B, where the hollow circles are.

Calculation of Balance. — Sometimes a tentative design for a rotating machine part is such that disturbance would clearly result if the part were made and run, and it may be desired to secure balance by adding or subtracting material. Let AB (Fig. 354) be the shaft of the proposed part,

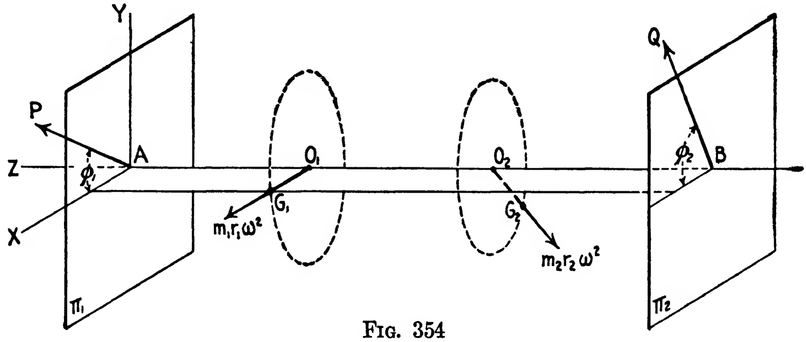


FIG. 354

π_1 and π_2 the planes in which the added (or subtracted) balancing masses M_1 and M_2, both unknown, are to rotate, and R_1 and R_2 (both unknown) the radiuses to *added* masses M_1 and M_2. The problem is to find (M_1R_1), (M_2R_2), and the angles ϕ_1 and ϕ_2, which R_1 and R_2 make with some reference plane including the axis of the shaft and fixed in the body.

We assume that the designed part consists of symmetrical portions, such as described in Art. 188 cases (1) or (2), for which the resultant effective forces are easily calculated. Let G_1, G_2, etc. be the mass-centers of the portions; O_1, O_2, etc. their centers of rotation; r_1, r_2, etc. the distances O_1G_1, O_2G_2, etc.; m_1, m_2, etc. the masses of the portions. The reversed effective forces for the portions are $m_1r_1\omega^2$, $m_2r_2\omega^2$, etc. where ω denotes any angular velocity; they are directed as shown in Fig. 354. The reversed effective forces for M_1 and M_2 respectively are

$$P = M_1R_1\omega^2 \quad \text{and} \quad Q = M_2R_2\omega^2.$$

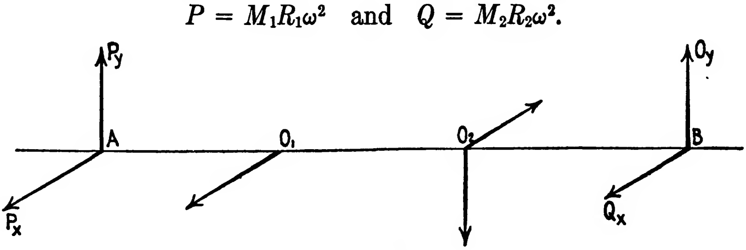


FIG. 355

In Fig. 355 we have shown all reversed effective forces (for the portions and the added balance masses) replaced by their x and y components. (The x-axis is taken for convenience parallel to O_1G_1.) The x components are in equilibrium; also the y components. Each system of components may be solved separately for its unknowns (P_x and Q_x, and P_y and Q_y

respectively); then P and Q and the angles ϕ_1 and ϕ_2 can be evaluated, all by obvious methods. Finally (M_1R_1) and (M_2R_2) can be computed from the equations above.

In each of these equations there are two unknowns (mass and radius). It is necessary to arbitrarily choose a value for one unknown and then solve for the other. (And since he may also take the planes π_1 and π_2 at pleasure, the designer has considerable leeway in the balance problem.) The angular velocity ω really cancels out from the foregoing equations because P and Q are proportional to ω^2. Hence balance is independent of speed, and in actual calculations ω^2 is omitted wherever it appears above. *But* if a body balanced as just explained be run at a speed so high as to *distort* the body, then it will be more or less out of balance.

Balancing Machines. — A rotating machine part which has been balanced "on paper" in the manner just explained is still apt to be somewhat out of balance on account of nonhomogeneity of material and inaccurate workmanship. If it is a high-speed part this residual balance may be serious and should be remedied. For this purpose there are now available excellent "balancing machines" by means of which an operator can detect residual unbalance and ascertain necessary corrections.* Formerly a simple but inadequate balancing machine was widely used. It merely consists of two accurately made horizontal parallel rails or ways. The part under examination or test is placed on the machine, the shaft across and resting on the rails. In general the part rolls from any random position, indicating that its center of gravity is not in the axis of rotation, and that the part is therefore out of balance. Addition or subtraction of material and repeated trying for roll eventually puts the center of gravity in the axis or nearly so; then the body is in "standing" or "static balance." But this operation merely eliminates the force R (Fig. 353) and leaves the couple F_1F_2.

191. Stresses in Moving Parts. — In Art. 61 stress is defined (for a truss member) as either of the forces which two parts of the member on opposite sides of any imagined cross section exert upon each other. Stresses occur not only when a member is subjected to axial end pulls or pushes, as in the case of a truss member, but also when it is bent or twisted, as in the case of beams or shafts. The definition of stress given above requires elaboration for these less simple cases, but for present purposes it is sufficient to remark that some sort of stress exists wherever one part of a body exerts a force upon a contiguous part.

The dynamic forces exerted by parts of a body upon one another constitute stresses, and these may be so great as to injure or disrupt the body. Thus an emery wheel may be rotated at so high a speed as to fly apart, or a piston rod moved back and forth so rapidly as to pull in two. D'Alem-

* See Journal Society of Automotive Engineers for October 1922 and April 1924, and American Machinist for Sept. 6, 1923.

bert's principle is helpful in determining the stresses that occur in moving bodies since it enables the body in question to be represented under a static loading (the reversed effective forces) that produces the same stresses as those that accompany the motion. In Ex. 1 below the application to a simple case of axial tension is illustrated. In Ex. 2 the principle is used to obtain the equivalent static loading on a moving member subject to bending.

EXAMPLE 1. Figure 356 represents diagrammatically the parts of a 3-inch field gun that recoil when the gun is fired. A is the gun proper (tube and jacket) and B is the recoil cylinder, attached to A by means of the rod C. When the gun is fired it slides back, carrying with it the recoil cylinder and forcing the oil with which the latter is filled to flow through small orifices in the stationary piston D. The resistance the oil offers to flow gives rise to a pressure, which, exerted against the forward end of the cylinder, constitutes the resistance to recoil that brings the gun to rest. At a particular

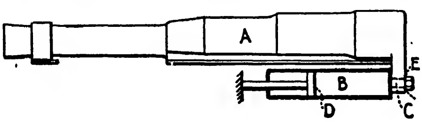


FIG. 356

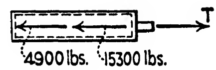


FIG. 357

instant the total backward powder pressure against the breech of the gun is 233,000 lbs. and the total forward oil pressure against the head of the cylinder is 4900 lbs. The gun is in a horizontal position, and the resistance offered to sliding by friction and other minor forces may be neglected. The weight of the gun is 835 lbs. and the weight of the recoil cylinder and rod C is 60 lbs. (Pressures and weights given are approximate.) It is required to determine the tension in the rod C at the instant in question.

Solution: The acceleration of the gun and cylinder is given by

$$\Sigma F_x = 233{,}000 - 4900 = \frac{895}{32.2}a, \text{ whence } a = 8210 \text{ ft/sec/sec.}$$

The recoil cylinder has motion of translation, with acceleration to the right equal to 8210 ft/sec/sec., therefore the effective system of forces for it reduces to a single force $R = (60/32.2)\,8210 = 15{,}300$ lbs.

Figure 357 represents the recoil cylinder and that part of the rod C to the left of section EE, with the resultant R of the effective system reversed and the horizontal external forces (tension in the rod T and forward pressure of the oil) represented; the vertical external forces (weight of cylinder and reaction of supports) are in equilibrium and need not be considered. The external forces and the reversed effective system being in equilibrium, we have

$$\Sigma F_x = T - 4900 - 15{,}300 = 0, \text{ whence } T = 20{,}200 \text{ lbs.}$$

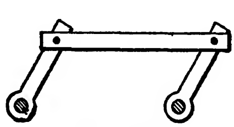


FIG. 358

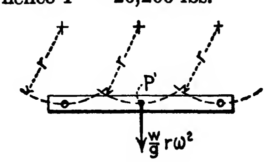


FIG. 359

EXAMPLE 2. Figure 358 represents a uniform bar pinned to two similar and parallel cranks, which rotate together with a constant angular velocity. The bar weighs w lbs. per foot of length; the crank length is r. It is evident that when at rest the bar will bend under its own weight; when in motion it will, part of the time, be bent even

more (this can be *perceived* by analogy with the case of a slender rod which is held at the ends in the hands and moved rapidly up and down). It is required to determine the increase in bending effect that accompanies the motion.

Solution: The bar has motion of translation, therefore the acceleration of all constituent particles is the same. Since each particle moves in a circle of radius r, its acceleration is $r\omega^2$, directed toward its center of rotation, and the effective force upon it is $mr\omega^2$, also directed toward its center of rotation. The effective system is therefore made up of the parallel forces $mr\omega^2$ acting on all the particles. The effective system reversed is in equilibrium with the external forces; obviously the bending effect of the combined systems is greatest when the reversed effective forces act in the same direction as the weights of the particles, which is the case when the bar is in its lowest position. For this position the reversed effective force $mr\omega^2$ on any particle P' is as shown in Fig. 359. Since all particles have the same acceleration, the resultant of the reversed effective system is $\Sigma mr\omega^2 = Mr\omega^2 = \frac{W}{g}r\omega^2$ for the whole rod, or $\frac{w}{g}r\omega^2$ for each foot thereof. The effect of the motion is therefore the same as that of increasing the weight of the bar $100\left(\frac{w}{g}r\omega^2 \div w\right) = \frac{100\,r\omega^2}{g}$ per cent. For the *highest* position of the bar, the effect is the same as that of applying an upward load (uniformly distributed) equal to $\frac{100\,r\omega^2}{g}$ per cent of the weight.

CHAPTER XII

WORK; POWER; ENERGY

§ 1. Work

192. Definition. — Work is a common word and has many meanings (see dictionary), but it is used in a single special sense in Mechanics. Work is said to be done upon a body by a force — also by the agent exerting the force — when the point of application of the force moves so that the force has a component along the path of the point of application.[1] This component will be called the *working component* of the force; and the length of the path of the point of application the *distance through which the force acts*. If the working component is constant the amount of work done by the force is taken as equal to the product of the magnitude of the working component and the distance through which the force acts. If the working component is not constant, the work is calculated as explained in the next article. Since it is the product of two scalar quantities, work is a scalar quantity.

The *unit work* is the work done by a force whose working component equals unit force acting through unit distance.[2] The unit of work depends therefore upon the units used for force and distance; thus we have the foot-pound, the centimeter-dyne (called also erg), etc. The horse-power-hour and the watt-hour are larger units of work. They are the amounts of work done in one hour at the rates of one horse-power and one watt, respectively (see Art. 201); thus,

$$\text{One horse-power-hour} = 1{,}980{,}000 \text{ foot-pounds, and}$$
$$\text{One watt-hour} = 36{,}000{,}000{,}000 \text{ ergs} = 3600 \text{ joules.}$$

When the works done by several forces are under discussion, signs should be given to their works according to this commonly used rule: When the working component acts in the direction of motion, the work of the force is regarded as positive; when the working component acts oppositely to the direction of motion, the work of the force is regarded as negative.

[1] Since the force which does work must be exerted by some *thing* (here called the agent) it is correct to say that work is done by the agent; thus a horse does the work of drawing a cart, a spring does the work of closing a door, etc. The amount of work done in any given case is usually determined by separately calculating the work done by each of the forces that act, and so in this discussion we usually speak of the work done by a force, rather than the work done by a body. But it is essential to keep in mind that all forces are exerted, and all work is done, by *things*.

[2] For dimensions of unit work, see Appendix A.

Forces which do positive work are sometimes called *efforts;* those which do negative work *resistances.*[1]

193. Calculation of Work Done by a Force. — The determination of the work done by a force is simple when the working component of the force is constant. For example suppose that the body represented in Fig. 360 is moved along the line AB by a number of forces, two of which (indicated) are constant in magnitude and in direction. During any portion of the motion, as from A to B, the work done by F_1 is $F_1\ (AB)$ and the work done by F_2 is $(F_2 \cos\theta)\ AB$. This expression when written $F(AB \cos\theta)$ means the product of the force and the component of the displacement along the line of action of the force, which is a "view" of amount of work done by a force sometimes more convenient than the other.

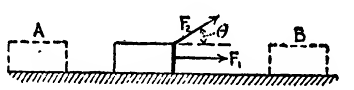


FIG. 360

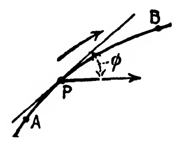


FIG. 361

When the working component is not constant in magnitude, then we may arrive at an expression for the work somewhat like this: — Let AB (Fig. 361) be the path of the point of application of one of the forces acting upon a body not shown, and P any point on the path. Let F = the force, ϕ = the angle between F and the tangent at P, and ds = the elementary portion of the path at P. Then the work done by F during the elementary displacement is $F \cos\phi \cdot ds$ or $F_t\, ds$ where F_t means working or tangential component of F; and the work done by F in the displacement from A to B is $\int F_t\, ds$, limits of integration to be assigned so as to include all elementary works $F_t ds$ in the motion from A to B. It is worth noting that if the force F acts normally to the path at all points, then $F_t = 0$ always, and the formula gives zero for the work done by F, as it should.

The formula $\int F \cos\phi\ ds$, with the lower and upper limits of integration to correspond to the initial and final positions A and B, respectively, observes this rule of signs for work, if s is measured positive in the direction of motion from some fixed origin to P, and ϕ is measured from the "positive tangent" around to the line of action of the force as shown in the figure.

EXAMPLE 1. The block B (Fig. 362) moves up the inclined plane under the action of certain forces, three of which are shown. F_1 is horizontal and equal to 20 lbs., F_2 is normal to the plane and equal to 10 lbs., F_3 is parallel to the plane and equal to 30 lbs.

[1] The (negative) work done by a resistance on a body is often referred to as (positive) work done by the body against the resistance.

It is required to determine the work done by each of these forces while the block moves 100 ft. up the plane.

Solution: The working component of F_1 is constant and is $20 \times \cos 30° = 17.3$ lbs; it acts in the direction of the motion, and so the work F_1 does is $+(17.3 \times 100) = 1730$

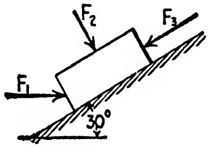


FIG. 362

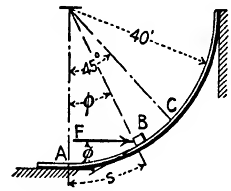


FIG. 363

ft-lbs. The working component of F_2 is zero; and so F_2 does no work. The working component of F_3 is 30 lbs.; it acts opposite to the direction of the motion and so the work F_3 does is $-(30 \times 100) = -3000$ ft-lbs.

EXAMPLE 2. Figure 363 represents a side view of a smooth track which is curved upward to form the arc of a circle of 40 ft. radius. A small body B weighing 10 lbs. is pushed along this track by a horizontal force F, the magnitude of which is at all times just sufficient to keep the body moving. It is required to determine the work done by F in pushing the body from A to C.

Solution: The work done by F is given by $\int F \cos \phi \, ds$. Both F and ϕ are variable; if they are expressed in terms of s the integral may be evaluated. To find F, the free body diagram for B is drawn and the forces acting solved for. Since B moves with uniform speed, $a_t = 0$ and so

$$\Sigma F_t = F \cos \phi - 10 \sin \phi = 0, \text{ whence } F = 10 \tan \phi.$$

Now from the figure it is evident that $\phi = s/40$; therefore the work done by F is

$$\int_0^{31.4} \left(10 \tan \frac{s}{40}\right)\left(\cos \frac{s}{40}\right) ds = 10 \int_0^{31.4} \sin \frac{s}{40} \, ds = 117.2 \text{ ft-lbs.}$$

194. Work Diagram.

— If values of F_t and s be plotted on two rectangular axes (Fig. 364) for all positions of the point of application of F, then the curve joining the consecutive plotted points might be called a "working force-space" (F_t-s) curve. The portion of the diagram "under the curve" (between the curve, the s-axis, and any two ordinates) is called the *work diagram* for the force F for the displacement corresponding to the bounding ordinates. The area of a work diagram represents the work done by the force during the displacement corresponding to the bounding ordinates. Proof: Let m = the force scale-number, and n = the space scale-number; that is, unit ordinate (inch) = m units of F_t (pounds) and unit abscissa (inch) = n units of s (feet). Also let A = area; then

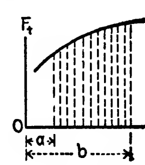


FIG. 364

$$A = \int_{x_1}^{x_2} y \, dx = \int_a^b \frac{F_t ds}{m \; n} = \frac{1}{mn} \int_a^b F_t ds = \frac{\text{work}}{mn}.$$

Hence, $A \times (mn)$ = work; that is, A = work according to the scale number mn to be used for interpreting the area.

By *average working component* of F is meant a value of F_t which multiplied by the distance $s_2 - s_1$, or $b - a$, gives the work done by F. Obviously, that average working force is represented by the average ordinate to the curve of the work diagram. When that curve is straight, that is, when F_t varies uniformly with respect to s, then the average working component equals the mean of the initial and final values.

Figure 365 is a fac-simile of a record made by the traction dynamometer (a spring balance essentially) in a certain train test. Abscissas represent distances traveled by the train, and ordinates represent "draw-bar pulls" (the pulls between the tender and first car of the train). Thus, the figure is a work diagram. To determine the area of such a diagram as this we first draw in an average curve "by eye," and then ascertain the area under this curve in any convenient way.

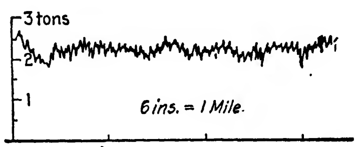


FIG. 365

EXAMPLE. — Refer to Fig. 363, and imagine a force F acting on the small body B always vertically (upwards) but changing in magnitude, and that its magnitude is known only at the lowest position of B and at 10° intervals as B moves along the arc, as recorded after F below. The work done by this variable force for the motion through 90° is required.

Solution: In the third line of the schedule we have recorded values of the angle $\theta = 90° - \phi$ between F and the direction of the motion,

ϕ =	0	10	20	30	40	50	60	70	80	90	deg.
F =	30	46	54	59	61	60	55	45	30	7	lbs.
θ =	90	80	70	60	50	40	30	20	10	0	deg.
F_t =	0	8.0	18.3	29.5	39.2	46.0	47.7	42.3	29.5	7.0	lbs.

and in the fourth line, values of the working component F_t. We might have drawn a work diagram for F_t, and then ascertained the required work from its area. Instead we found the arithmetic mean (26.75 lbs.) of the ten values of F_t, which is a close approximation of the true average F_t. The product of the average and the distance through which the force acts 62.8 ft.), gives the work, or 1680 ft-lbs.

195. Some Important Special Cases. — Many of the practical problems met with which involve the determination of work done come under one or another of the following special cases: (i) Work done by a force constant in direction and magnitude; (ii) Work done by gravity; (iii) Work done by a number of forces with a common point of application; (iv) Work done by a force always directed through a fixed point; (v) Work done by two equal, opposite and colinear forces.

These five special cases will now be discussed in the order of naming.

196. Work Done by a Force Constant in Direction and Magnitude. — The work done by a force which is constant in magnitude and direction

equals the product of the force and the projection of the displacement of its point of application upon the line of action of the force. For, let F = the force, APB (Fig. 366) the path of its point of application, ϕ = the (variable) angle between F and the direction of the motion of the point of application P. Then the work done by F is

$$\int F \cos \phi \, ds = F \int ds \cos \phi,$$

where ds is an elementary portion of the path. Now $ds \cos \phi$ is the projection of the element ds upon F, or upon any line parallel to F, and $\int ds \cos \phi$ is the sum of the projections of all the elements of APB upon the line. But the sum of the projections = the projection of APB = the projection of the chord AB.

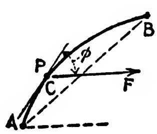


FIG. 366

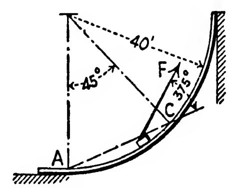


FIG. 367

EXAMPLE. The body B (Fig. 367) is drawn up the circular track by a force F, which is always equal to 12 lbs. and always acts at an angle of 60° to the horizontal. It is required to determine the work done by F while B moves from A to C.

Solution: Since F is constant in direction and magnitude, the work it does is equal to the product of its magnitude and the projection of the displacement AC on the line of action of F. $AC = 2 \times 40 \sin 22.5° = 30.6$ ft.; the projection of AC on the line of action of F is $30.6 \times \cos 37.5° = 24.3$ ft.; therefore the work done by F is $12 \times 24.3 = 292$ ft-lbs. Since the working component of F is directed like the motion, the work done is positive.

197. Work Done by Gravity. — The work done by gravity upon a body in any motion equals the product of its weight and the vertical distance described by the center of gravity; the work is positive or negative according as the center of gravity has descended or ascended. Let w_1, w_2, etc., denote the weights of the particles of the body; y_1', y_2', etc., their distances above some datum plane — below which the body does not descend — at the beginning of motion; and y_1'', y_2'', etc., their distances above that plane at the end of the motion (see Fig. 368, where $a'a''$ is the path of the first particle, $b'b''$ that of the second, etc.). Also let W denote the weight of the body, and $\bar{y}'$ and $\bar{y}''$ the initial and final heights of its center of gravity above the plane. Then the works done by gravity on the particles respectively, are

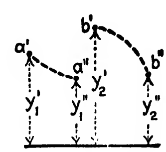


FIG. 368

$w_1(y_1' - y_1'')$, $w_2(y_2' - y_2'')$, etc., and the sum of these works can be written

$$(w_1y_1' + w_2y_2' + \cdots) - (w_1y_1'' + w_2y_2'' + \cdots).$$

The first term of this sum $= W\bar{y}'$, and the second $= W\bar{y}''$ (see Art. 89); hence the sum of these works done on all the particles equals

$$W\bar{y}' - W\bar{y}'' = W(\bar{y}' - \bar{y}'').$$

EXAMPLE. A cylindrical cistern 20 ft. deep and 6 ft. diameter, containing 10 ft. of water, is pumped dry, the water being discharged at the ground level. It is required to determine the work done by gravity on the water.

Solution: The weight of the water is 17,650 lbs; the vertical displacement of its center of gravity is 15 ft.; therefore the work done by gravity is $-(17{,}650 \times 15) = -265{,}000$ ft-lbs., negative because the center of gravity has ascended. (The work done "against gravity" is 265,000 ft-lbs.)

198. Work Done by a Number of Forces with a Common Point of Application. — The algebraic sum of the works done by any number of forces having a common point of application during any displacement of that point equals the work done by their resultant during that displacement. For, let F', F'', etc., = the forces, R = their resultant, and F_t', F_t'', etc., and R_t = the components of the forces and of the resultant, respectively, along the tangent to the path of the point of application. Now $F_t' + F_t'' + \cdots = R_t$ (Art. 26). Hence $F_t'\,ds + F_t''\,ds + \cdots = R_t\,ds$, and

$$\int F_t'\,ds + \int F_t''\,ds + \cdots = \int R_t\,ds;$$

that is, the sum of the works done by the forces equals the work done by their resultant.

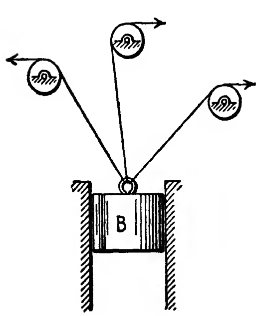


FIG. 369

EXAMPLE. Figure 369 represents a body B, weighing 50 lbs., which is raised vertically at a uniform rate by means of three ropes arranged as shown. It is required to determine the work done collectively by the three ropes on the body while the latter is raised 4 ft.

Solution: Since the body moves uniformly it is in equilibrium under its own weight, the horizontal reactions of the (smooth) guides, and the three rope pulls. Therefore the resultant of the three rope pulls is an upward force equal to 50 lbs., and the work this resultant does is $50 \times 4 = 200$ ft-lbs. This is also the work done collectively by the three ropes on the body B.

199. Work Done by a Force Always Directed Through a Fixed Point. — The work done by a force always directed through a fixed point is equal to

$$+\int_{r_1}^{r_2} F\,dr \quad \text{or} \quad -\int_{r_1}^{r_2} F\,dr$$

according as the force tends to move its point of application away from or towards the fixed point; here F = magnitude of the force (not constant

necessarily), r = the distance between its point of application and the fixed point, and r_1 and r_2 are respectively the initial and final values of r.

Let O, Fig. 370, be the fixed point and F the force, which moves its point of application P along the path shown. The work done by F during an elementary displacement ds is $F \cos \phi \, ds$. But it is obvious from the figure that $\cos \phi \, ds = dr$; hence the work done by F in any elementary displacement is $F\,dr$, and the total work is given by $\int F\,dr$. Obviously, changing the sense of F changes the sign of the work done.

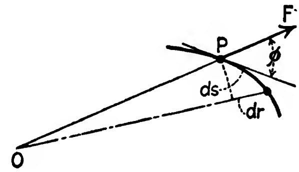


FIG. 370

EXAMPLE 1. A body B weighing 200 lbs. is dragged up an irregular slope (Fig. 371) by means of a cable and winch as indicated. A constant tension of 250 lbs. is maintained in the cable. It is required to determine the work done by the cable on the body in hauling it from the bottom to the top of the slope.

Solution: The pull F of the cable acts (practically) through the fixed point O (top of sheave), and the work it does is

$$-\int_{r_1}^{r_2} F\,dr = -\int_{110}^{15.6} 250\,dr = 23{,}600 \text{ ft-lbs.}$$

EXAMPLE 2. Figure 372 represents a wire bent into a circular arc and a bead B which slides on the wire. The bead is connected to the fixed point O by a rubber cord, the normal length of which is 1 ft. and which stretches $\frac{1}{2}$ inch for each pound of tension.

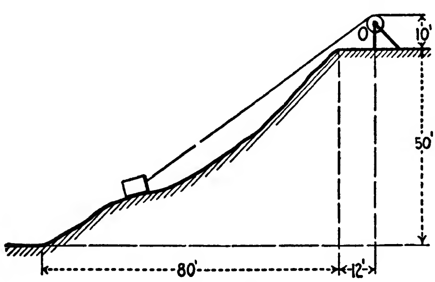


FIG. 371

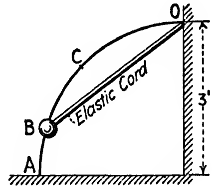


FIG. 372

The bead is released from point A whence it moves up the wire. It is required to determine the work done on the bead by the cord during the displacement A to C (halfway between A and O).

Solution: The cord exerts on the bead a variable force F which always acts through O. The magnitude of F is 2 lbs. for every inch, or 24 lbs. for every foot, the cord is stretched; therefore if r denote the distance in feet from B to O, $F = (r-1)\,24$, and the work done by F is

$$-\int_{r_1}^{r_2} F dr = -\int_{4.24}^{2.30} (r-1)\,24\,dr = 105 \text{ ft-lbs.}$$

200. Work Done by Two Equal and Opposite Forces. — The work done by a pair of equal, colinear, and opposite forces in any displacement of

their points of application equals

$$+\int_{r_1}^{r_2} F\,dr \quad \text{or} \quad -\int_{r_1}^{r_2} F\,dr$$

according as the forces tend to separate or draw the points of application together; here F = the common magnitude of the two forces (not constant necessarily), r = the distance between their points of application, and r_1 and r_2 = initial and final values of r. Let F_1 and F_2 (Fig. 373) represent the equal forces, P_1 and P_2 their points of application which move along the paths shown. The work done by F_1 in an elementary displacement ds_1 is $F \cos \phi_1 ds_1$; the work done by F_2 is $F \cos \phi_2 ds_2$; the work done by both forces is the sum of these quantities, or $F(\cos \phi_1 ds_1 + \cos \phi_2 ds_2)$. But it is obvious from the figure that $\cos \phi_1 ds_1 + \cos \phi_2 ds_2 = dr$, hence the work done by both forces in the elementary displacement is $F\,dr$, and the total work is given by $\int F\,dr$. Obviously, changing the senses of F_1 and F_2 changes the sign of the work.

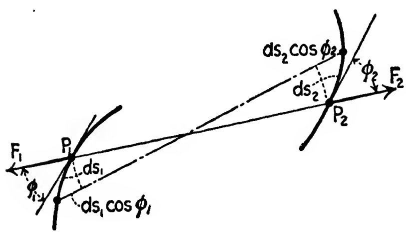


FIG. 373

EXAMPLE. Two balls are connected by a light spiral spring, the normal length of which is 2 ft. and which shortens 2 in. for each 1 lb. compression. The balls are pressed to within 6 in. of each other and the whole then tossed into the air. It is required to determine the work done by the spring on the balls whilst expanding to its normal length.

Solution: The spring exerts equal and opposite forces F on the two balls. Let r = length of spring, in inches, at any instant; then $F = \frac{1}{2}(24 - r)$ lbs., and the work done is

$$+\int_{r_1}^{r_2} F dr = +\int_{6}^{24} \tfrac{1}{2}(24 - r)\,dr = 81 \text{ in-lbs.}$$

Solution may be effected more simply by using the *average* value of F. Thus the maximum compression in the spring is $(24 - 6) \times \frac{1}{2} = 9$ lbs.; the average force it exerts is half this or 4.5 lbs.; the value of Δr is 18, and the work done is $4.5 \times 18 = 81$ in-lbs.

§ 2. Power

201. Definition. — The time rate at which work is done is called power. In common parlance the word power is used with several somewhat different meanings; thus the adjective powerful may mean, capable of exerting a great force, or, capable of doing a great amount of work, or, capable of working at a high rate. But in most engineering usage power refers specifically to one thing — *time rate of doing work*, and we shall here use it in that sense alone.

Units of Power[1] like units of work may be classed into gravitational, which vary slightly with locality, and absolute. Thus, the foot-pound per minute and the kilogram-meter per second are units of the first class; also the (practical)

$$\text{English and American horse-power} = 550 \text{ foot-pounds per second} = 33{,}000 \text{ foot-pounds per minute,}$$

and the

$$\text{Continental horse-power} = 75 \text{ kilogram-meters per second} = 4500 \text{ kilogram-meters per minute.}$$

The dyne-centimeter (or erg) per second is a unit of the second class; also the watt which is 10^7 ergs per second, and the (practical)

$$\text{kilowatt} = 1000 \text{ watts} = 10^{10} \text{ ergs per second.}$$

The Bureau of Standards has recently decided to adopt the English and American horse-power as the exact equivalent of 746 watts, thus making this horse-power an absolute unit. "Thus defined it is the rate of work expressed by 550 foot-pounds per second at 50° latitude and sea level, approximately the location of London, where the original experiments were made by James Watt to determine its value. The 'continental horsepower' is similarly most conveniently defined as 736 watts, equivalent to 75 kilogram-meters per second at latitude 52° 30′, or Berlin."[2]

202. Calculation of Power. — At times it is required to determine the power developed by a force or forces that act on a body (as a car or machine) and cause it to move in a particular manner. The *average* power for a given interval is the work done during that interval divided by the interval. The power developed at a particular instant might be determined by ascertaining the limiting value of the average power as the interval is taken smaller and smaller, but it is more readily determined as follows.

For any elementary period of time dt a given force F effects a displacement of its point of application ds, and the work done is $F_t\,ds$. Then $F_t\,ds/dt$ is the average power for the indefinitely short period dt — i.e., the power at the particular instant when dt is taken. But $F_t\,ds/dt = F_t v$, and so the power developed by a force at a particular instant is seen to be equal to the tangential component of the force times the speed with which its point of application is moving at that instant. The power developed by a system of forces is the algebraic sum of the powers developed by the several constituent forces, power being positive when the work being done is positive and negative when the work being done is negative.

Example 1. A block weighing 200 lbs. is moved along a horizontal floor at the rate of 30 ft/sec. (i) by means of a horizontal force F and (ii) by means of a force P inclined at 40° to the horizontal. Coefficient of friction is 0.3. It is required to determine the horse power developed by the moving force in each case.

[1] For dimensions of a unit of power see Appendix A.

[2] *Circular of the Bureau of Standards,* No. 34.

Solution: (i) The force F required to move the body is equal to the friction to be overcome, or $0.3 \times 200 = 60$ lbs. The power is equal to $60 \times 30 = 1800$ ft-lbs/sec. $= 1800 \div 550 = 3.27$ H.P. (ii) Calling the normal component of the floor reaction N, $\Sigma F_y = N - 200 + P \sin 40° = 0$ and $\Sigma F_x = P \cos 40° - 0.3\,N = 0$; solution gives $P = 62.5$ lbs. The working component of $P = P \cos 40° = 47.9$ lbs.; the power is $47.9 \times 30 = 1440$ ft-lbs/sec. $= 2.62$ H.P.

It appears from the results that while a greater force is required in the second case, the *work* done, for any given displacement, is less.

EXAMPLE 2. A body weighing W falls freely without air resistance. It is required to derive a general formula for the power developed by the weight of the body.

Solution: Dating time from the instant the body is released, the velocity of the mass-center of the body is given by $v = gt$. The force W is therefore at any instant doing work at the rate of Wgt. If W is in pounds, t in seconds, and g in feet per second per second, then the rate is in foot-pounds per second.

203. Measurement of Power. — To determine the power which a completed machine, as a gas engine, hydraulic turbine or dynamo, is capable of developing, engineers have recourse to direct experiment and measurement. Electrically, power may be measured directly by means of an instrument called the watt-meter. Mechanically, power is measured indirectly by means of devices which determine the work done in a given time; the work done divided by the time gives the average power. The work done may be conveniently measured by ascertaining the torque against which a rotating part turns and the rate of rotation. The torque can be regarded as due to a tangential force F acting at the end of an arm r. The work done by this force per revolution would then be $F \times 2\,\pi r$, and if n represents the number of rotations in unit time, the work done in unit time, that is the (average) power, would be $2\,\pi Frn$ or $2\,\pi Tn$, where T is the torque against which the rotation is maintained, easily measured by any of several devices. The Prony brake, widely used for determining power, operates on the principle here described.

§ 3. Energy

204. Definitions. — When the state or condition of a body is such that it can do work, the body is said to possess energy. A body may have energy by virtue of its motion; such energy is called *kinetic* energy. A body may also have energy by virtue of some circumstances other than motion, as position, configuration, state of stress, etc.; such energy is called *potential* energy.

The amount of energy a body has is equal to the amount of work the body can do, and so is measured in work units. Like work, energy is a scalar quantity.

205. Standard Conditions. — A body at an elevation above the earth has potential energy due to its position, and in descending can be made to raise another body by means of ropes and pulleys, thus doing work. A compressed spring has potential energy due to its state of stress, and

can lengthen against resistance, thus doing work. A falling pile driver has kinetic energy due to its motion, and can drive a pile into the ground, thus doing work. Now in each of these cases the body in question does work only by *changing* from one position, configuration or velocity to some other position, configuration or velocity, and the amount of work it can do in so changing is definite only when some specific final condition is taken as standard. The standard position may conveniently be taken as at the earth's surface; the potential energy due to position is then the work a body can do in descending to the surface of the earth. The standard configuration may conveniently be taken as that in which there is no internal stress, and the potential energy of configuration is then the work a body can do in straightening out (if bent) or in expanding (if compressed) or in shortening (if stretched). The standard velocity may be conveniently taken as zero (with respect to the earth), and the kinetic energy of a body is then the work it can do in coming to rest. We shall proceed to evaluate kinetic energy on this basis, first for a particle, subsequently for a rigid body having various motions.

206. Kinetic Energy of a Particle. — Consider the particle P (Fig. 374) which has, when at A, the velocity v_1. It moves along the path AB, and is brought to rest at B, under the action of the forces F_1, F_2, F_3, etc. We are to find the work done by the particle in giving up its velocity. The work done by the forces F_1, F_2, F_3, etc., on the particle is

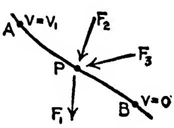


FIG. 374

$$\int F_1 \cos\phi_1\, ds + \int F_2 \cos\phi_2\, ds + \cdots = \int (F_1 \cos\phi_1 + F_2 \cos\phi_2 + \cdots)\, ds,$$

where ϕ_1, ϕ_2, ϕ_3, etc., are the angles between F_1, F_2, F_3, etc., and the direction of motion (Art. 193). Since the particle P exerts forces on its neighbors, equal and opposite to F_1, F_2, etc., the work done by the particle on its neighbors is

$$-\int (F_1 \cos\phi_1 + F_2 \cos\phi_2 + \cdots)\, ds.$$

But $F_1 \cos\phi_1 + F_2 \cos\phi_2 + \cdots = ma_t = m\, dv/dt$, where a_t is the tangential component of the acceleration of the particle; hence the work done by P is

$$-\int_0^{s'} m\,(dv/dt)\, ds = -\int_{v_1}^{0} m\,(ds/dt)\, dv = -\int_{v_1}^{0} mv\, dv = \tfrac{1}{2}\, mv_1^2.$$

The kinetic energy of a body (a collection or system of particles) is the sum of the kinetic energies of the constituent particles of the body.

207. Kinetic Energy of a Rigid Body in Translation. — In translatory motion all particles of the moving body have at each instant equal velocities; hence, the sum of the kinetic energies of the particles is $\frac{1}{2}\, m_1 v^2 +$

$\frac{1}{2} m_2 v^2 + \cdots = \frac{1}{2} v^2 (\Sigma m)$, where m_1, m_2, etc., = the masses of the particles and v = their common velocity at the instant under consideration. Or, if M = the mass of the body and E = energy, then

$$E = \frac{1}{2} Mv^2 = \frac{1}{2} (W/g)v^2.$$

If 32.2 is written for g, then v should be expressed in feet per second. E will be in foot-pounds, foot-tons, etc., according as W is expressed in pounds, or tons, etc.

EXAMPLE. A slender bar (Fig. 375) is mounted on two pivoted arms of equal length, which are rotated together at the (constant) rate of 40 rev/min. The rod is 3 ft. long and weighs 12 lbs.; the arms are 4 ft. long. It is required to determine the kinetic energy of the bar.

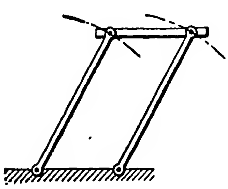


FIG. 375

Solution: The bar has motion of translation; its velocity is $v = (2\ \pi\ 4)\ 40/60 = 16.8$ ft/sec. Its kinetic energy is therefore given by $E = \frac{1}{2}\ (12/32.2)\ (16.8)^2 = 52.5$ ft-lbs.

208. Kinetic Energy of a Rigid Body in Rotation. — In a rotation about a fixed axis the velocity of any particle of the body equals the product of the angular velocity of the body, expressed in radians per unit time, and the distance from the particle to the axis of rotation (Art. 138). Hence, the sum of the kinetic energies of the particles of the body is

$$\frac{1}{2} m_1 (r_1\omega)^2 + \frac{1}{2} m_2 (r_2\omega)^2 + \cdots = \frac{1}{2} \omega^2 \Sigma mr^2,$$

where ω = the angular velocity of the body at the instant under consideration, and r_1, r_2, etc., = the distances of the particles respectively from the axis of rotation. But Σmr^2 = the moment of inertia of the body about the axis of rotation; hence, the kinetic energy is given by

$$E = \frac{1}{2} I\omega^2 = \frac{1}{2} Mk^2\omega^2 = \frac{1}{2} (W/g)k^2\omega^2,$$

where I = the moment of inertia described, and k = the radius of gyration of the body about the axis of rotation.

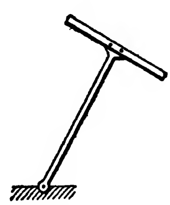


FIG. 376

EXAMPLE. The bar described in the example of Art. 207 is rigidly attached at its center to a pivoted arm 4 ft. long as shown in Fig. 376; the whole is then rotated at the (constant) rate of 40 rev/min. It is required to determine the kinetic energy of the bar.

Solution: The bar has motion of rotation with an angular velocity $\omega = 2\ \pi \times 40/60 = 4.18$ rad/sec. Its moment of inertia about the axis of rotation is

$$I = \frac{1}{12}\left(\frac{12}{32.2}\right) 3^2 + \left(\frac{12}{32.2}\right) 4^2 = 6.25 \text{ sl-ft.}^2$$

(see Appendix B). Its kinetic energy is

$$E = \frac{1}{2}\ (6.25)\ (4.18^2) = 54.6 \text{ ft-lbs.}$$

209. Kinetic Energy of a Rigid Body in Plane Motion. — Let Fig. 377 represent the moving body and O its mass-center; let M be the mass of the body, I its moment of inertia about a line through the mass-center perpendicular to the plane of motion, k its radius of gyration about the same line, $\bar{v}$ the velocity of the mass-center, and ω the angular velocity of the body. Also let P represent any elementary particle of the body and let m be the mass of P, v its velocity, and r its distance from the line through O perpendicular to the plane of motion. Then v is the resultant of $\bar{v}$ and $r\omega$ as indicated. The angle $QPS = 90 - (\beta - \theta)$, where β and θ are the angles which v and OP respectively make with the x-axis. Therefore

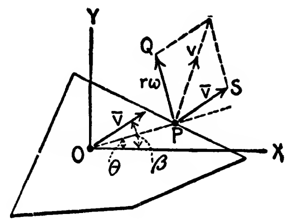


FIG. 377

$$v^2 = \bar{v}^2 + r^2\omega^2 + 2\,\bar{v}r\omega \sin(\beta - \theta),$$

and the kinetic energy of the entire body ($\Sigma\,\frac{1}{2}\,mv^2$) equals

$$\tfrac{1}{2}\,\bar{v}^2\Sigma m + \tfrac{1}{2}\,\omega^2\Sigma mr^2 + 2\,\bar{v}\omega\,(\sin\beta\Sigma mr\cos\theta - \cos\beta\Sigma mr\sin\theta).$$

Now $r\cos\theta$ and $r\sin\theta$, respectively, equal the x and y coördinate of P. Hence $\Sigma mr\cos\theta = \Sigma mx = \bar{x}\Sigma m$ (see page 199), $\bar{x}$ denoting the x coördinate of the mass-center; and since $\bar{x} = 0$, $\Sigma mr\cos\theta = 0$. Similarly, $\Sigma mr\sin\theta = 0$. Hence the foregoing expression for the kinetic energy reduces to

$$\tfrac{1}{2}\,\bar{v}^2\Sigma m + \tfrac{1}{2}\,\omega^2\Sigma mr^2 = \tfrac{1}{2}\,M\bar{v}^2 + \tfrac{1}{2}\,I\omega^2.$$

The first term of this equation equals the kinetic energy which the body would have if its motion were one of translation with velocity equal to $\bar{v}$; and the second term equals the kinetic energy which it would have if its motion were one of rotation about a fixed axis through the mass-center and perpendicular to the plane of the motion. Hence the kinetic energy of a body with any plane motion may be regarded as consisting of two parts; they are called translational and rotational, and each is quite independent of the other.

EXAMPLE. A cylindrical disc 6 ft. in diameter weighing 400 lbs. rolls so that its center has a velocity of 4 ft/sec. It is required to determine the kinetic energy of the disc. *Solution:* The moment of inertia of the disc about an axis through its mass-center perpendicular to the plane of motion is $I = \frac{1}{2}\left(\frac{400}{32.2}\right)3^2 = 55.9$ sl-ft.^{2} and its angular velocity is $v/r = 4/3$ rad/sec. Its kinetic energy therefore is

$$E = \tfrac{1}{2}\left(\frac{400}{32.2}\right)(4)^2 + \tfrac{1}{2}\,(55.9)\left(\frac{4}{3}\right)^2 = 99.4 + 49.7 = 149 \text{ ft-lbs.}$$

210. Kinetic Energy of a Rigid Body Having Spherical Motion. — As in Art. 149 let ω_x, ω_y and ω_z = the axial components of the angular velocity of the moving body at any particular instant; x, y and z = the coördinates

of some particle P of the body then; and v_x, v_y and v_z = the axial components of the velocity of P. If m = the mass of the particle, then its kinetic energy at the instant in question is

$$\tfrac{1}{2}\, m\, (v_x^2 + v_y^2 + v_z^2).$$

Now if we substitute for v_x, v_y and v_z their values from the equations of Art. 149, we arrive at a new expression for the kinetic energy of P; and if we sum up such expressions for all the particles of the body, we find that the kinetic energy of the body is

$$\tfrac{1}{2}\, I_x\omega_x^2 + \tfrac{1}{2}\, I_y\omega_y^2 + \tfrac{1}{2}\, I_z\omega_z^2 - K_x\omega_y\omega_z - K_y\omega_z\omega_x - K_z\omega_x\omega_y,$$

where I_x, I_y and I_z are the moments of inertia of the moving body about the x, y and z axes respectively, and K_x, K_y and K_z are the products of inertia of the body with respect to the pairs of coördinate planes intersecting in the x, y and z axes respectively (Art. 188) all at the instant in question. That is

$$K_x = \Sigma myz, \quad K_y = \Sigma mzx, \quad K_z = \Sigma mxy.$$

The products of inertia may be zero; then the kinetic energy equals

$$\tfrac{1}{2}\, I_x\omega_x^2 + \tfrac{1}{2}\, I_y\omega_y^2 + \tfrac{1}{2}\, I_z\omega_z^2.$$

211. Kinetic Energy of a Body Having General Motion. — Let O be the center of mass of the body. We will regard the motion as consisting of a translation like the motion of O, and a rotation about some line through O. Let $\bar{v}_x$, $\bar{v}_y$ and $\bar{v}_z$ be the axial components of the velocity of O; ω_x, ω_y and ω_z the axial components of the angular velocity about O. Then the components of the velocity of any particle P are (see Art. 151)

$$\begin{aligned} v_x &= \bar{v}_x + z'\omega_y - y'\omega_z, \\ v_y &= \bar{v}_y + x'\omega_z - z'\omega_x, \\ v_z &= \bar{v}_z + y'\omega_x - x'\omega_y. \end{aligned}$$

Since the kinetic energy of the particle is $\tfrac{1}{2}\, mv^2 = \tfrac{1}{2}\, m(v_x^2 + v_y^2 + v_z^2)$, the kinetic energy of the body is $\Sigma\, \tfrac{1}{2}\, mv^2$, or

$$\tfrac{1}{2} M\, (\bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2) + \tfrac{1}{2}\, I_x\omega_x^2 + \tfrac{1}{2} I_y\omega_y^2 + \tfrac{1}{2} I_z\omega_z^2 - K_x\omega_y\omega_z - K_y\omega_z\omega_x - K_z\omega_x\omega_y.$$

Now the first term of this expression equals $\tfrac{1}{2}\, M\bar{v}^2$, and is the value which the kinetic energy would have if all the material were concentrated at the mass-center and were moving with it. The remainder of the expression is the value which the kinetic energy of the body would have if the center of mass were fixed (Art. 210).

§ 4. Principles of Work and Kinetic Energy

212. For a Particle. — In any displacement of a particle the forces acting on it, if any, do work; in general the velocity of the particle changes during the displacement, and hence its kinetic energy changes also. It will now be shown that *in any displacement of a particle the work done by all the*

forces acting on it equals the increment in the kinetic energy of the particle. Let P (Fig. 378) be the particle; m = its mass; OAB be its path (not a plane curve necessarily); v_1 = its velocity at A, v_2 = its velocity at B; R = the resultant of all the forces acting on P; and R_t = the component of R along the tangent to the path at P. Then the work done by all the forces during an elementary displacement ds is $R_t\,ds$. But $R_t = ma_t = m\,dv/dt$, where a = tangential component of the acceleration of P. Hence the work done on P in the displacement ds is $m(dv/dt)ds = m(ds/dt)dv = mv\,dv$; and the work done in the total displacement AB is

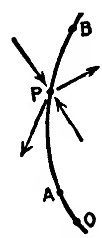


FIG. 378

$$\int_{v_1}^{v_2} mv\,dv = \tfrac{1}{2}\,mv_2^2 - \tfrac{1}{2}\,mv_1^2.$$

Now $\frac{1}{2}\,mv_2^2$ is the kinetic energy of the particle at B, and $\frac{1}{2}\,mv_1^2$ is its kinetic energy at A; hence $\frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2$ is the increment in the kinetic energy of P. And so the equality stated above is proved. If the total work done on the particle is positive then the increment in the kinetic energy is positive also, and there is a real gain in velocity; if the total work done on the particle is negative then the increment in the kinetic energy is negative also, and there is a loss in velocity.

213. For a Body. — Let P_1, P_2, P_3, etc., be the particles of any body (not rigid necessarily). In any displacement of the body,

work done by forces acting upon P_1	=	increment in kinetic energy of P_1,
" " " P_2	=	" " " P_2,
" " " P_3	=	" " " P_3,
etc.	=	etc.

Adding we get total work done on all particles = sum of increments in their kinetic energies = increment in kinetic energy of the system. That is, *in any displacement of any body the total work done upon it by all the external and internal forces acting upon it equals the increment in the kinetic energy of the body.*

In a displacement of a rigid body the total work done by the internal forces equals zero. Proof: — Consider any internal force exerted, say, on P_1 by P_2; P_1 exerts an equal, opposite, and colinear force on P_2. Since the body is rigid the distance between the points of application (P_1 and P_2) of these two forces does not change, and hence (Art. 200) the total work done by these two forces equals zero. But all the internal forces occur in such pairs; hence, the total work done by all the internal forces equals zero, as stated. Thus we have the principle, — *in any displacement of a rigid body the total work done upon the body by the external forces acting upon it equals the increment in the kinetic energy of the body.*

From the relations stated above it follows that the rate at which work

is done upon a body equals the rate at which it gains kinetic energy. But the rate at which work is done is power, and so *the combined power of all the forces doing work upon a body at any instant equals the rate at which it is gaining kinetic energy then.*

214. Typical Problems. Examples. — The principles of work and kinetic energy are especially well adapted for ascertaining the change in the velocity of a body during any displacement for which it is possible to compute the total work done on the body. By their application we can also ascertain something about the forces and displacements which accompany any given change in the kinetic energy of a body.

EXAMPLE 1. A block weighing 50 lbs. is pushed for a distance of 30 ft. along a horizontal floor by a constant horizontal force of 25 lbs. The coefficient of friction between block and floor is 0.2. It is required to determine the resulting velocity of the block, and also to determine how far the block would slide before coming to rest if the push ceased to act after the 30-ft. displacement.

Solution: The forces acting on the block while the displacement is occurring are its own weight, the 25-lb. push, the normal component of the floor reaction $N = 50$ lbs., and the friction component of the floor reaction $0.2\ N = 10$ lbs. The weight and the normal component of the floor reaction do no work; the work done by the applied push is equal to $25 \times 30 = 750$ ft-lbs. (positive); the work done by the friction component of the floor reaction is equal to $10 \times 30 = 300$ ft-lbs. (negative). Since the body is rigid the internal forces do no work, and so the total work done is $750 - 300 = 450$ ft-lbs. This is equal to the increment in the kinetic energy of the block, or

$$450 = \frac{1}{2}\frac{50}{32.2}v^2, \quad \text{whence} \quad v = 24.1 \text{ ft/sec.}$$

To determine how far the block would slide after the push ceased to act we set the total work done during the entire displacement equal to zero (since there is no increment in the kinetic energy in the entire displacement). Let x = the distance the block thus slides; while the block undergoes the displacement x the only force that does work on it is friction, and this work is equal to $-10\,x$. Therefore, for the entire displacement the total work done is

$$450 - 10\,x = 0; \quad \text{whence} \quad x = 45 \text{ ft.}$$

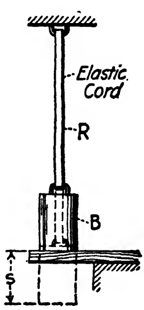


FIG. 379

EXAMPLE 2. A rubber cord R (or other elastic body) is attached at one end to a fixed support and at the other to a body B, weight W (Fig. 379). The cord is of such a character that its stretch is e when B hangs freely and at rest. It is required to determine the tension caused in the cord by suddenly removing the support under B and allowing it to fall until stopped by the tension in the cord.

Solution: Let s = the distance dropped through by B at any instant; then s is also the stretch in the cord, and the tension in the cord is $T = (W/e)s$. The forces that do work on B during its descent are T and its own weight W. The total amount of work done on B up to the time B is brought to rest is zero (increment in kinetic energy is zero); therefore if x denote the total stretch of the cord

$$+Wx - \int_0^x \frac{W}{e}s\,ds = 0, \quad \text{whence} \quad Wx = \frac{1}{2}\frac{W}{e}x^2, \quad \text{or} \quad x = 2\,e.$$

That is the stretch of the cord when fully extended is twice the stretch when the load B hangs freely and at rest. Hence the tension in the cord when fully extended is $2\,W$. Thus, it is seen that the effect of applying the load *suddenly* (but without initial velocity) is to cause twice the stress in the supporting member that would exist under ordinary static conditions.

EXAMPLE 3. A certain flywheel and its shaft weigh 400 lbs.; the radius of gyration of both with respect to the axis of rotation = 10 in. The wheel is set to rotating at 100 rev. per min., and is then left to itself, coming to rest under the influence of axle friction and air resistance after making 84 turns. It is required to determine the average torque of the resistances.

Solution: The moment of inertia of the wheel and shaft about the axis = $(400/32.2)(10/12)^2 = 8.64$ sl-ft.². The angular velocity, 100 rev/min. $= 2\,\pi\,100/60 = 10.47$ rad/sec. Hence, the kinetic energy of this wheel and shaft, when released, $= \frac{1}{2}\,8.64 \times 10.47^2 = 474$ ft-lbs. Besides the forces mentioned above, gravity and the normal pressure of the bearings act on the wheel and shaft, but these do no work during the stoppage. Let T = average torque of the resistances in ft-lbs.; then the work done by them during the stoppage is $-T\,2\,\pi\,84 = -528\,T$ ft-lbs. This equals the change in the kinetic energy of the wheel; therefore

$$528\,T = 474, \quad \text{whence} \quad T = 0.898 \text{ ft-lbs.}$$

EXAMPLE 4. A slender uniform rod of length l is pinned at one end to a horizontal floor. The rod is raised to a vertical position, and then released and allowed to fall over. It is required to determine the speed with which the free end will be moving when it strikes the floor.

Solution: Neglecting air resistance and friction on the pin, the only work done on the rod as it falls is that done by gravity, equal to $\frac{1}{2}\,Wl$. The bar has motion of rotation, therefore its kinetic energy when it strikes the floor is

$$E = \tfrac{1}{2}\,I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{12}\frac{W}{g}l^2 + \frac{W}{g}\tfrac{1}{4}l^2\right)\omega^2 = \tfrac{1}{6}\frac{W}{g}l^2\omega^2.$$

But $\omega = v/l$, where v = speed of the end of the rod, and so

$$\tfrac{1}{2}\,Wl = \tfrac{1}{6}\frac{W}{g}v^2, \text{ whence } \quad v = \sqrt{3\,gl}.$$

EXAMPLE 5. A (Fig. 380) is a sheave supported on a rough horizontal shaft. A is 3 ft. in diameter, and its radius of gyration with respect to the axis of rotation = 9 in. The weights of A, B and C are 100, 200 and 300 lbs., respectively. The system is released and allowed to move under the influence of gravity and the resistances brought into action. It is required to determine the velocity of the suspended weights when they have moved through 10 ft.

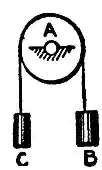


FIG. 380

Solution: The system moves under the action of the following external forces, — gravity, axle reaction, air resistance, and the internal reactions between sheave and rope and the fibers of the rope. If the rope is quite flexible then the forces in the rope do little work; this will be neglected. If the rope does not slip on the sheave, then no work is done by the reaction between rope and sheave. Thus, little or no work is done by the internal forces. The work done by air resistance is small unless the speeds of the moving bodies get high; it will be neglected. The work done by the frictional component of the axle reaction per turn is $T\,2\,\pi$, where T is the frictional moment which we will assume has been found to be 10 in-lbs. In the displacement under consideration, 10 ft. for B and C, the wheel makes $10/3\,\pi$ turns. Hence, the total work done by friction $= (10 \times 2\,\pi)(10/3\,\pi) = 66.7$ in-lbs. $= 5.6$ ft-lbs. Gravity does no work on A; on B and C its work $= 300 \times 10 - 200 \times 10 = 1000$ ft-lbs.; its work on the rope is neglected as small.

Hence, the total work done on the system = 1000 − 5.6 = 994.4 ft-lbs. Now let v = the required velocity in ft/sec.; then the angular velocity of the wheel = $v \div 1.5$ = $0.667\,v$ rad/sec. The kinetic energy of the system equals

$$\frac{1}{2}\frac{300}{32.2}v^2 + \frac{1}{2}\frac{200}{32.2}v^2 + \tfrac{1}{2}I\,(0.667\,v)^2,$$

where I = moment of inertia of the sheave. Now $I = (100/32.2) \times (9/12)^2 = 1.75$ sl-ft.^{2}; hence, the kinetic energy of the system = $8.16\,v^2$ ft-lbs. Therefore the work-energy equation is

$$994.4 = 8.16\,v^2, \quad \text{whence} \quad v = 11 \text{ ft/sec.}$$

EXAMPLE 6. A certain pair of car wheels and their axle weigh 2000 lbs. Their diameter is 33 in. and the radius of gyration of wheels and axle is 9 in. They are rolled along a level track until their speed is 60 rev/min., and are then left under the influence of the rolling resistance of the track, coming to rest after rolling a distance of 1000 ft. (Data not from an actual experiment.) It is required to determine the average rolling resistance.

Solution: When released, the angular velocity of the wheels = 1 rev/sec. = 6.28 rad/sec., and the linear velocity of their centers = $\pi\,33/12$ = 8.64 ft/sec. Hence, the kinetic energy is

$$\frac{1}{2}\frac{2000}{32.2} \times 8.64^2 + \frac{1}{2}\frac{2000}{32.2}(9/12)^2 \times 6.28^2 = 3010 \text{ ft-lbs.}$$

This is also the value of the work done by the rolling resistance, air resistance neglected. Hence, the rolling resistance is equivalent to a constant pull-back of 3010/1000 = 3 lbs.

§ 5. Kinetic Friction; Efficiency

215. Kinetic Friction. — Kinetic Friction, or Friction of Motion, is the friction between two bodies when sliding actually occurs. The *coefficient of kinetic friction* for two bodies is the ratio of the kinetic friction to the corresponding normal pressure between them. The *angle of kinetic friction* is the angle between the normal pressure and the total pressure (resultant of the normal pressure and the kinetic friction). It is commonly assumed that the kinetic coefficient of friction is less than the static coefficient (Art. 73), the implication being that there is a sudden or abrupt change in the values of the coefficients. Experiments at speeds as low as 0.0002 ft/sec. have been made that indicate that the kinetic coefficient of friction gradually increases when the velocity becomes extremely small, so as to pass without discontinuity into the static coefficient. Under certain conditions the kinetic coefficient may exceed the static coefficient, and in general it varies, for given materials, not only with the velocity but also with the intensity of pressure and with the length of time that rubbing continues. Thus for metals the coefficient appears to decrease as the intensity of pressure increases until seizing occurs, while continual rubbing of dry surfaces abrades them and decreases the coefficient of friction.

Experimentally determined values of the coefficient of kinetic friction for certain materials and conditions are given in the following table.

COEFFICIENTS OF KINETIC FRICTION (Rough Averages)
Compiled by Rankine from Experiments by Morin and others

Wood on wood, dry	0.25–0.50	Leather on oak	0.27–0.38
soapy	.2	Leather on metals, dry	.56
Metals on oak, dry	.5– .60	wet	.36
wet	.24– .26	greasy	.23
soapy	.2	oily	.15
Metals on elm, dry	.2– .25	Metals on metals, dry	.15–0.2
Hemp on oak, dry	.53	wet	.3?
wet	.33		

216. Friction in Journals and Bearings. — So-called *coefficients of journal friction* have been determined from direct experiments on journal friction. This coefficient is the ratio of the frictional resistance to the pressure between the journal and the bearing.

The pressure between a journal and its bearing is not uniformly distributed over the surface of contact. By nominal intensity of pressure ("pressure" for brevity) is meant the whole pressure divided by the product of the length and diameter of the bearing.

It has been found from numerous experiments that coefficients of journal friction depend on (i) the method of lubrication, (ii) the lubricant, (iii) its temperature, (iv) the velocity of rubbing, and (v) intensity of pressure on the bearing.

The way in which the coefficient was found to vary with temperature, velocity and pressure in one series of experiments is shown by the curves of Fig. 381, 382 and 383.[1] In these tests forced lubrication was employed

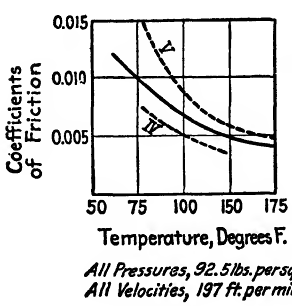


FIG. 381

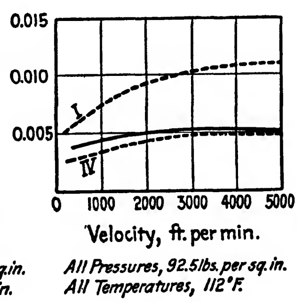


FIG. 382

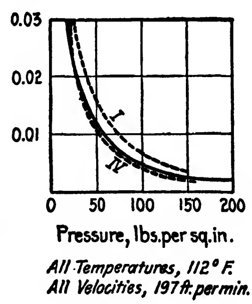


FIG. 383

and the journal and bearing combinations were as given in the following table.

Number	I	II	III	IV	V
Journal	steel	nickel steel	nickel steel	nickel steel	wrought iron
Bearing	white metal	white metal	....	bronze	white metal

[1] Lasche. *Z. d V. d. I* **1902**, Vol. 46, p. 1881.

The heavy line in each figure represents the average law for the five combinations, and the other two curves relate to the two combinations departing most from the average result.

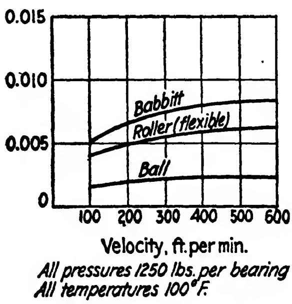


FIG. 384

Roller and Ball Bearings generally develop less resistance to turning than ordinary bearings. Fig. 384 gives coefficients of friction for, and hence a comparison of, such bearings for a $2\frac{7}{16}$ inch line shaft.[1]

217. Vehicle Resistance. — Any moving simple vehicle, like a horse-drawn wagon, experiences axle resistance, rolling resistance (Art. 84) and, unless the speed is quite low, air resistance. There is abundant information available as to the first named resistance, but little in regard to the other two. But their total or aggregate has been determined for many conditions or circumstances. The total of the first two we shall call "vehicle resistance." It may be regarded as a single imaginary horizontal backward force, applied to the vehicle, which is equivalent to the axle and rolling resistance.

Vehicle resistance obviously depends much on the kind of roadway, the tire, and the axle. It is generally believed to vary about as the weight of the vehicle including load if any. Thus this resistance is generally expressed in pounds per ton (of weight).

Vehicle Resistance for Ordinary Farm Wagons[2]

Kind of Road	*Resistance*
Earth roads — ordinary conditions	50–200 lb/ton
Gravel roads	50–100
Concrete, unsurfaced	27– 30
Brick	15– 40

Of more general interest nowadays is the resistance of automobiles. Fig. 385[3] gives their combined vehicle and air resistance, determined mostly from experiments with trucks. (Friction of the transmission system is not included.)

Vehicle resistance at low speed can be determined by dragging the vehicle as a trailer at constant speed and then measuring the pull required to drag. The vehicle resistance is equal to this pull. (If the vehicle is an automobile, the transmission is to be disconnected so that it does not

[1] Thomas, Maurer, and Kelso, *Jour. Am. Soc. Mech. Engrs.* for March, 1914.

[2] From Baker's *Roads and Pavements*, which includes a digest of many experimental determinations of resistance of horse-drawn vehicles.

[3] A portion of a figure taken from *Bulletin No.* 67, Iowa State College Engineering Experiment Station; but the word Vehicle in Fig. 385 replaces the word Rolling in the original figure in conformity with our usage of the words.

operate.) This method is simple in principle but the test is not easily conducted because the pull cannot be maintained steady.

The method used to determine the results shown in Fig. 385 was as follows: The automobile was dragged as a trailer, and then cast loose and allowed to coast on the level roadway. From a space-time record made by suitable apparatus in the automobile under test, the velocity time graph was made from which the acceleration of the automobile was calculated at various times during the coast. Then the total resistance acting at any particular time or speed was calculated from

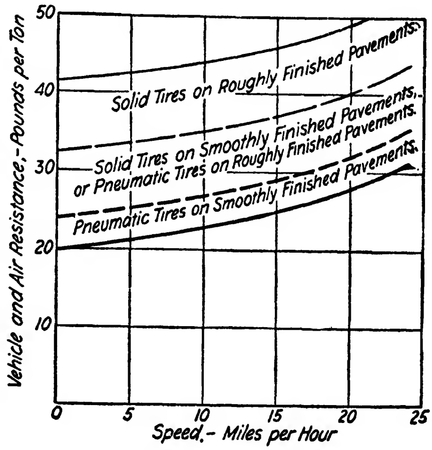


FIG. 385

$$R = Ma + 4\,I\frac{\alpha}{r} = \left(M + 4\,m\frac{k^2}{r^2}\right)a,$$

where M = mass of the whole car, I = moment of inertia of one wheel about its axis, r = radius of the wheels; a = linear acceleration of the car at the time and α = the angular acceleration of the wheels; m = mass of one wheel; and k = radius of gyration of a wheel about its axis. (If the wheels are not all alike, then *average* values must be used for I, m and k^2.)

EXAMPLE. It is required to derive the preceding formula.
Solution: At any instant the kinetic energy of the car is

$$\tfrac{1}{2}\,Mv^2 + 4 \times \tfrac{1}{2}\,Iw^2$$

where v = velocity of the car and ω = angular velocity of the wheel. On account of the resistance this kinetic energy decreases during the coast. The rate at which the decrease occurs is equal to the rate at which the total resistance R does work. That is

$$\frac{d}{dt}\left(\tfrac{1}{2}\,Mv^2 + 4 \times \tfrac{1}{2}\,I\omega^2\right) = Rv,$$

or

$$Mv\frac{dv}{dt} + 4\,I\omega\frac{d\omega}{dt} = Rv; \quad \text{whence}$$

$$R = M\frac{dv}{dt} + 4\,I\frac{\omega}{v}\frac{d\omega}{dt} = Ma + 4\,I\frac{\alpha}{r} = \text{etc.}$$

218. Efficiency of Machines. — Among the machines and appliances used in the industries there are some whose function is the conversion or transmission of energy. For example, — an electric dynamo which converts mechanical into electrical energy; a steam engine which converts energy of steam into (kinetic) energy of its flywheel; a line-shaft which transmits energy from one place in a shop to one or more other places;

etc. In this article, "machine" means the kind of machine or appliance just described. The amount of energy supplied to a machine in any interval of time, for conversion or transmission, is called the *input* for that time; the amount of energy converted into the desired form or transmitted to the desired place is called the *output*. Experience has shown that output is always less than input; that is, a machine does not convert or transmit the entire input. The difference between output and input, for the same interval of time of course, is called lost energy or *loss* simply. By *efficiency*, in this connection, is meant the ratio of output to input; that is, if e = efficiency, then

$$e = output \div input.$$

Most machines are designed for a definite rate of working or for a certain load called full load. Then we speak of the efficiency of a machine at full load, half-load, quarter over-load, etc., these efficiencies being different generally. The two following tables are given to furnish some notion of the efficiencies of the more common machines.[1]

FULL-LOAD EFFICIENCY OF	Per cent
Hydraulic turbines	60–85
impulse wheels	75–85
Steam boilers	50–75
engines	5–20
turbines	5–20
Gas and oil engines	16–30
Electric dynamos	80–92
motors	75–90
transformers	50–95

EFFICIENCY OF SOME MACHINE ELEMENTS*	Per cent
Common bearing, singly	96–98
Common bearing, long lines of shafting	95
Roller bearings	98
Ball bearings	99
Spur gear cast teeth, including bearings	93
Spur gear cut teeth, including bearings	96
Bevel gear cast teeth, including bearings	92
Bevel gear cut teeth, including bearings	95
Belting	96–98
Pin-connected chains, as used on bicycles	95–97
High-grade transmission chains	97–99

* From Kimball and Barr's *Elements of Machine Design*.

The efficiency of a combination of machines, A, B, C, etc., A transmitting to B, B to C, etc., is the product of the efficiencies of the individual machines. For, let e_1, e_2, e_3, etc. = the efficiencies of the separate machines A, B, C, etc., and e = the efficiency of the group. Then if E = the input for A, the output of $A = e_1E$ = the input for B; the output of $B = e_2e_1E$ = the input for C; the output of $C = e_3e_2e_1E$ = the input for D; etc. Hence, the output of the last machine $\div$ the input of the first $= (e_1e_2e_3 \cdots) E \div E = e_1e_2e_3 \cdots$ or,

$$e = e_1 \cdot e_2 \cdot e_3 \cdots.$$

[1] For detailed information see Mead's *Water Power Engineering*, from which most values in the first table were taken; Gebhardt's *Steam Power Plant Engineering;* and Franklin and Esty's *Elements of Electrical Engineering*.

For example, if a dynamo is run by a steam engine, then the efficiency of the combination or set = the product of their separate efficiencies, say $0.20 \times 0.90 = 0.18$ or 18 per cent.

219. Hoisting Appliances. — There are certain rather simple appliances by means of which a given force can overcome a relatively large resistance; as, for example, the lever, the wedge, the screw, the pulley, etc. Such an appliance is generally operated by means of a single force, which we call *driving force* and denote by F; it is called effort also. The force which the appliance is desired to overcome we call resisting force and denote by R, or when the force is a weight, by W; it is called resistance also. In many appliances (hereafter called "*common*") all equal displacements of the point of application of the driving force result in equal displacements of the point of application of the resisting force; and generally these displacements respectively take place along the lines of action of the driving and resisting forces. These displacements, or their components along the lines of action of the forces respectively if the displacements are inclined to the forces, we will denote by a and b respectively.

In a common appliance, the input (work done by the driving force F) and the output (work done by the appliance against the resisting force R) are respectively Fa and Rb; hence the efficiency is given by

$$e = Rb \div Fa. \quad \text{(1)}$$

Let F_0 = the effort which would be required to overcome the resistance R if the machine were frictionless; then $F_0a = Rb$. Substituting in (1) we find that efficiency is given also by

$$e = F_0 \div F. \quad \text{(1')}$$

Let R_0 = the resistance which F could overcome if the machine were frictionless; then $Fa = R_0b$. Substituting in (1) we find that efficiency is given also by

$$e = R \div R_0. \quad \text{(1'')}$$

Most of the appliances now under discussion can be operated backward as well as directly. For example, the lever, the wedge, the screw, etc., can be used to lower a heavy body as well as to raise it. Some of these appliances, which can be run either way, will run backward without direct assistance when loaded; that is the load will overcome the internal friction. Such appliances are said to *overhaul.* Some will not run backward unassisted; that is, the load cannot overcome the internal friction. Such appliances are said to *self-lock* (see Art. 79). An appliance will overhaul or self-lock according as its (direct)[1] efficiency is greater or less than one-

[1] When a machine is run backwards it is said to have *reversed efficiency,* by (considerable) extension of the definition of efficiency. In such case the load (on the hoist, for example) is the effort, and the applied force is regarded as the useful resistance. In case the machine self-locks so that the applied force (P, say) must assist the load, then by considerable stretch of imagination $-P$ is regarded as the useful resistance; the computed (reversed) efficiency is negative.

half, if the works done in overcoming friction in a forward and in an equal backward motion are equal (usual case). Proof: — As before, let F = the effort, R = the (useful) resistance, a and b = corresponding displacements of F and R, and w = the work done against friction, all in forward motion of the appliance. Then $Fa = Rb + w$. Now if the efficiency (forward motion) is greater than one-half, then more than one-half of the work Fa is expended usefully (against R); that is, Rb is greater than w, and hence R could overcome the friction in backward motion. If the efficiency (forward motion) is less than one-half, then less one-half the work Fa is done against R; that is Rb is less than w, and hence R could not overcome friction unassisted in backward motion.

By *mechanical advantage* of an appliance is meant the ratio of the resisting to the driving force when the appliance is operating steadily, at constant speed. Thus, see equation (1), mechanical advantage is given by

$$R/F = ea/b. \quad \ldots \ldots \ldots \ldots \quad (2)$$

Obviously the value of the ratio a/b does not depend on the loss or wasted work; that is, it is independent of the efficiency. (The ratio depends solely on the geometrical proportions of the appliance.)[1] Hence we may assume $e = 1$ and write $a/b = R'/F'$ where R'/F' means the mechanical advantage of the appliance if it were without friction. Finally,

$$R/F = eR'/F' \quad \ldots \ldots \ldots \ldots \quad (2')$$

or, (mechanical advantage) = (efficiency e) $\times$ (mechanical advantage at $e = 1$).

In some appliances or mechanisms the driving and (useful) resisting forces (F and R) are applied at a wheel (pulley, gear, etc.), and it is more convenient in the discussion to deal with the torques, or moments of the forces about the shaft axes respectively, than with the forces. Let T_1 and T_2 denote those torques, of the driving and resisting forces respectively, and α and β corresponding angular displacements, in radians, of the wheels to which T_1 and T_2 are applied. The works done by the force F and against R during the displacements α and β are $T_1\alpha$ and $T_2\beta$. Hence, the efficiency is $e = T_2\beta/T_1\alpha$, and

$$T_2/T_1 = e\alpha/\beta. \quad \ldots \ldots \ldots \ldots \quad (3)$$

Reasoning as in the preceding paragraph we conclude that $\alpha/\beta = T_2'/T_1'$ where T_2'/T_1' means the ratio of the resisting torque to the driving torque if the appliance were frictionless, $e = 1$. Hence

$$T_2/T_1 = eT_2'/T_1'. \quad \ldots \ldots \ldots \ldots \quad (3')$$

[1] When the displacements a and b are not inclined to the forces F and R respectively, then the ratio a/b is sometimes called the *velocity ratio* of the appliance, for the velocities of the points of application of F and R are as a to b. Thus we have for such cases

mechanical advantage = efficiency $\times$ velocity ratio.

High mechanical advantage requires high velocity ratio, b small compared to a; thus the adage "what is gained in force is lost in velocity."

We may call the ratio of the resisting and driving torques, the mechanical advantage of torque; then the foregoing result may be stated as follows:

(mechanical advantage of torque) = (efficiency e) $\times$ (mechanical advantage of torque at $e = 1$).

A simple pulley, with part of a rope or chain upon it, is represented in Fig. 386. Let S = tension in leading or off side of the rope, and T = tension in the following or on side. Then evidently S is greater than T, for S overcomes not only T but also the friction at the pin and the "rigidity" of the rope. The resistance due to pin friction = the product of the coefficient of axle friction (see Art. 216) and the pressure on the pin; this pressure = $S + T$. Hence, if f = coefficient and r = radius of the pin, the work done against friction per revolution of the pulley = $2\pi rf(S + T)$. The work done in bending or unbending (inelastic) rope over the pulley is proportional to the amount of rope so bent per revolution (that is $2\pi R$), and it seems to be proportional also to the tension, to the area of the cross section of the rope, and inversely proportional to the radius R. Thus the work of bending = $C\,2\pi RT\,d^2/R = C\,2\pi T\,d^2$, where d = diameter of the rope and C is an experimental coefficient depending on the kind of rope and perhaps other elements.

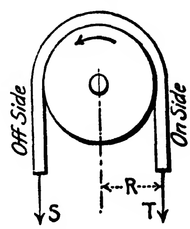


FIG. 386

Likewise the work of unbending (at off side) = $C\,2\pi S\,d^2$. Now, if we equate the work done by the effort S to the work done against rigidity, the resistance T, and the axle friction, and then simplify the resulting equation, we get

$$S\left(1 - f\frac{r}{R} - C\frac{d^2}{R}\right) = T\left(1 + f\frac{r}{R} + C\frac{d^2}{R}\right).$$

This equation can be written in the following approximately correct form, —

$$S = \left(1 + 2f\frac{r}{R} + 2\,C\frac{d^2}{R}\right)T = KT,$$

where K is an abbreviation for $1 + 2fr/R + 2\,C\,d^2/R$. According to experiments by Eytelwein, C equals about 0.23 when d and R are expressed in inches. The American Bridge Company made some experiments to determine C and K for such pulleys and rope as are in common use in tackle for construction work, and found that C depends not only on kind of rope, as expected, but also on the size of rope. The following table is taken from their report.[1]

[1] *Trans. Am. Soc. C. E.*, 1903, Vol. 51, p. 161; also *Eng. Rec.*, 1903, Vol. 48, p. 307.

Dimensions and coefficients	Hemp				Wire
Diameter of rope, d	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$\frac{3}{4}$
Center pin to center rope, R	$3\frac{7}{8}$	$4\frac{9}{16}$	$5\frac{3}{8}$	6	$7\frac{3}{8}$
Diameter of pin, $2r$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{3}{8}$	$2\frac{1}{2}$
Coefficient C	0.23	0.20	0.19	0.17	0.9
Coefficient K	1.20	1.21	1.23	1.24	1.16

These values of C and K are higher than those usually employed. $C = 0.08$ to 0.22 is advised for hemp rope,[1] and "$K = 1.06$ to 1.07 may be considered maximum practical values since generally $K = 1.02$ to 1.04."[2]

Tackle refers to an assemblage of ropes and pulleys, usually so arranged as to give some desired mechanical advantage. The mechanical advantage of any such combination can be expressed in terms of K, as shown below for the three particular cases represented in Fig. 387, 388 and 389. Let W be the load to be lifted or lowered, as the case may be, and let P be the corresponding effort required. Then:

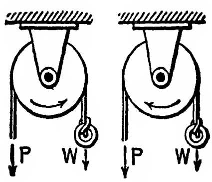


FIG. 387

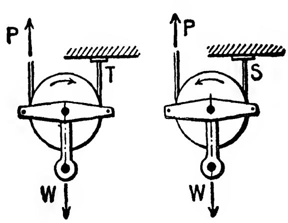


FIG. 388

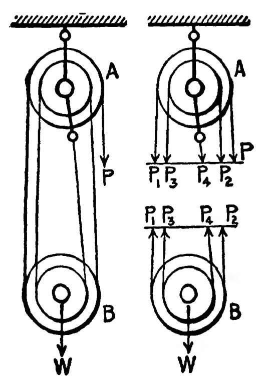


FIG. 389

(i) When a fixed pulley (Fig. 387) is used for lifting,

$$P = KW, \text{ whence } W/P = 1/K,$$

and when used for lowering,

$$W = PK, \text{ whence } W/P = K.$$

(ii) When a movable pulley (Fig. 388) is used for lifting,

$$W = P + T = P + P/K, \text{ whence } W/P = (1 + K)/K,$$

and when used for lowering,

$$W = P + S = P + KP, \text{ whence } W/P = 1 + K.$$

(iii) When the tackle shown in Fig. 389 (consisting of two separate and similar pulleys in each block, represented as of different sizes for clearness) is used for lifting,

$$P_1 = P/K, P_2 = P_1/K, \text{ etc.},$$

[1] Hütte, *Taschenbuch* (Twentieth Edition), Vol. 1, p. 247.

[2] Bottcher-Tolhausen, *Cranes*, p. 15.

and $\quad W = P_1 + P_2 + P_3 + P_4$, whence

$$W/P = (K^3 + K^2 + K + 1)/K^4,$$

and when used for lowering,

$$P_1 = KP, P_2 = KP_1, \text{ etc.}$$

and $\quad W = P_1 + P_2 + P_3 + P_4$, whence

$$W/P = K(1 + K + K^2 + K^3).$$

Special (Chain) Hoists. — Figure 390 represents a Weston differential hoist. The upper block contains two pulleys differing slightly in diameter; they are fastened together. The lower block contains only one pulley. The pulley grooves have pockets into which the links of the chain fit; thus slipping of the chain is prevented. The chain is endless and is reeved as shown. If there were no lost work, then the tension in each portion of the chain to block B would equal one-half the load, and the pulls on the block A would be as indicated in the figure. Now if R and r = the distances from the center of the pin in block A to the axis of the chain as indicated then moments about the axis of the pin give

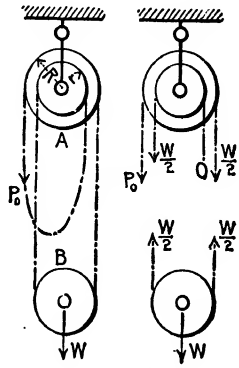


FIG. 390

$$P_0R + \tfrac{1}{2} Wr = \tfrac{1}{2} WR, \quad \text{or} \quad W = P_0 2R/(R - r);$$

the ratio, $W/P_0 = 2R/(R - r)$ may be made very large by making $R - r$ small. The mechanical advantage is

$$\frac{W}{P} = \frac{W}{P_0}e = \frac{2R}{R - r}e,$$

where P = the actual force required to raise W and e = efficiency. These hoists are made of various capacities up to W = 3 tons; their efficiencies are relatively low, from about 25 to 40 per cent according to the manufacturers' lists. In the so-called Duplex and Triplex hoists the upper blocks are screw-geared and spur-geared respectively. At full load the efficiency of these hoists varies from about 30 to 40 and from 70 to 80 per cent.

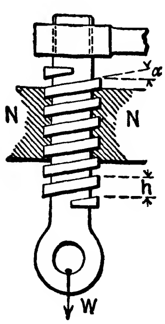


FIG. 391

EXAMPLE 1. The pitch of the screw-jack (Fig. 391) is h, the mean radius of the screw thread is r, the length of the lever is l. It is required to determine the efficiency of the jack when it is overcoming (raising) W.

Solution: It is shown in Art. 80 that the force required at the end of the lever to start the screw is $P = W\,(r/l) \tan(\phi + \alpha)$, where ϕ is the angle of static friction and α is the pitch angle, $\tan^{-1}(h/2\pi r)$. Hence the force required to raise the load W uniformly is given by the same expression if ϕ is taken to denote the angle of kinetic friction (see Art. 215). Let it be so understood in what follows. If the screw were frictionless then the force required would be $P_0 = W(r/l) \tan \alpha$; and hence the efficiency is $e = \tan \alpha \div \tan(\phi + \alpha)$.

EXAMPLE 2. Figure 392 represents a double purchase crab, for hoisting. Hand cranks can be applied on the ends of the shaft B or C. The hoisting rope winds on the drum. The crank is 18 in. long; the drum is 10 and the rope $\frac{3}{4}$ in. in diameter; gears A' and B are 20 and B' and C are 4 in. in diameter. It is required to determine the mechanical advantage of the appliance when a crank is used on C.

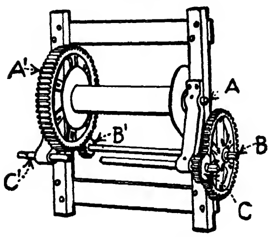


FIG. 392

Solution: Evidently, angular displacements of gears A' and B' respectively are as 1 to 5; also the angular displacements of gears B and C. Hence, for one turn of A', C makes $5 \times 5 = 25$ turns. For one turn of A', $b = 10.75\,\pi$, and a (the corresponding displacement of the point of application of the effort) $= 36\,\pi \times 25 = 900\,\pi$. Therefore, the velocity ratio of the appliance is 83.8. If the efficiency is 80 per cent say, the mechanical advantage is $0.80 \times 83.8 = 67$.

CHAPTER XIII

MOMENTUM AND IMPULSE

§ 1. Linear Momentum and Impulse

220. (Linear) Momentum. — By momentum of a moving particle is meant the product of its mass and velocity. Momentum is regarded as having direction, namely that of velocity; thus, momentum is a vector quantity. By momentum of a collection of particles or body is meant the vector-sum of the momentums of the particles. For example, let m' and m'' = the masses of two particles (Fig. 393), v' and v'' = the velocities of the particles at a certain instant, and suppose that $AB = m'v'$ and $BC = m''v''$ according to some convenient scale; then AC represents the momentum of the two particles.

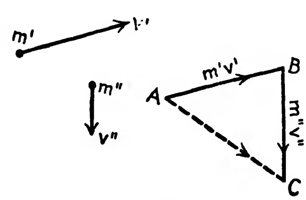


Fig. 393

In the case of a body having a motion of translation, all the particles have at any instant velocities which are equal in magnitude and the same in direction (Art. 133). Hence the momentums of the particles are parallel, and their vector sum is

$$m'v + m''v + \ldots = v\Sigma m = Mv,$$

where v = their common velocity and M = the mass of the body.

The *unit of momentum* is the momentum of a body of unit mass moving with unit velocity. The magnitude of the unit, therefore, depends on the units of mass and velocity used. No single word has been generally accepted for any unit of momentum. The dimensional formula for momentum is FT (see Appendix A), that is, a unit momentum is one dimension in force and one in time. Hence, any unit of momentum may be and commonly is called by names of the units of force and time used. Thus the unit of momentum in the C.G.S. system is called the dyne-second; in the "engineers' system," the pound (force) -second; etc.

221. Component of Linear Momentum. — Since the component of the vector AC along any line equals the algebraic sum of the components of the vectors AB and BC along that line, it follows that the component of the momentum of a pair of particles along any line equals the algebraic sum of the components of their momentums along that line. Obviously, this proposition can be extended to a collection of any number of particles. A simple expression for this component can be arrived at as follows:

Let m', m'', etc. = the masses of the particles; v', v'', etc. = their velocities; and v'_x, v''_x, etc. = the components of these velocities along any line x. Then the component of the momentum of the collection along this line $= m'v'_x + m''v''_x + \cdots$ Now if x', x'', etc. = the x coördinates of the moving particles, and $\bar{x}$ = the x coördinate of the mass-center, all at the same instant, then

$$m'x' + m''x'' + \cdots = \bar{x}\Sigma m \text{ (Art. 172)};$$

and differentiating with respect to t, we get

$$m'dx'/dt + m''dx''/dt + \cdots = (d\bar{x}/dt)\Sigma m,$$

or $$m'v'_x + m''v''_x + \cdots = \bar{v}_x\Sigma m = M\bar{v}_x,$$

where $M = \Sigma m$ = the mass of the collection. That is, the x component of the momentum of the collection of particles equals the product of the mass of the collection and the x component of the velocity of the mass-center. Hence, the component momentum is just the same as though all the material of the body were concentrated at the mass-center.

222. Principle of Force and Linear Momentum. — *The component along any line of the external system of forces acting on any body is equal to the rate at which the linear momentum of the body along that line is changing.* For, from Art. 172,

$$\Sigma F_x = M\bar{a}_x; \text{ and since } \frac{d}{dt}(M\bar{v}_x) = M\frac{d\bar{v}_x}{dt} = M\bar{a}_x,$$

$$\Sigma F_x = \frac{d}{dt}(M\bar{v}_x).$$

The principle just arrived at was derived from the principle of motion of the mass-center (Art. 173), and is essentially an alternative form of the law. But practically the former seems to apply more simply in certain cases as the following examples show.

EXAMPLE 1. Figure 394 represents a jet of water impinging against a flat plate. The mass of the water impinging per unit time is M; the velocity of the jet is v; and the inclination of the jet to the plate is α as shown. Required the pressure of the jet upon the plate.

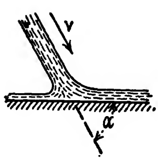


FIG. 394

Solution; We suppose that the water does not rebound from the plate with any considerable velocity; then the momentum of the water after striking has no component normal to the plate. The momentum of an amount of water equal to M before striking is Mv, and the component of that momentum along the normal to the plate $= Mv \sin\alpha$; hence the change in the (normal) component momentum is $Mv \sin \alpha$. This change takes place in unit time; therefore, it is the rate at which momentum along the normal is changed, and also the value of the normal pressure of the plate against the jet. The jet exerts an equal (normal) pressure against the plate. If the plate is rough, then the water also exerts a frictional force on the plate.

EXAMPLE 2. Figure 395 (left) represents a bend in a water pipe. The mass of water flowing past any section of the pipe per unit time is M; the velocity of the water, assumed to be the same at all points of inlet and outlet cross sections of the bend, is v; and α is the angle of the bend as shown. Required the resultant pressure of the flowing water against the bend.

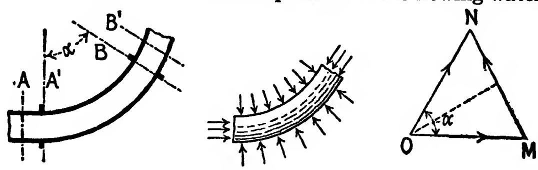


FIG. 395

Solution: Let Δt = the time required for the body of water AB to move into the position $A'B'$. The momentum of the body of water at the beginning of the interval = that of AA' + that of $A'B$; its momentum at the end of the interval = that of $A'B$ + that of BB'. Hence the change in the momentum of the body of water in the time Δt = momentum of BB' − momentum of AA'. These momentums respectively are in the direction BB' and AA'; each equals $(M\Delta t)v$. Hence the change of momentum under consideration is represented by the vector MN where OM and ON represent the two momentums just mentioned. But $MN = 2\,(OM) \sin \frac{1}{2}\alpha$; hence the change $= 2\,(M\Delta t)v \sin \frac{1}{2}\alpha$, and the rate at which the change occurs $= 2\,Mv \sin \frac{1}{2}\alpha$. The direction of this rate is MN; it bisects the angle α. This rate of change of momentum is maintained by the forces acting on the body of water in $A'B$. Those forces consist of gravity G, the pressures P_1 and P_2 (of the water) on the front and rear faces of the body, and the pressure P of the bend upon it. Their resultant $R = 2\,Mv \sin \frac{1}{2}\alpha$, and R bisects α. If R, G, P_1 and P_2 are known then P can be determined. For it is such a force as compounded with G, P_1 and P_2 gives R. The pressure of the water on the bend $= -P$.

EXAMPLE 3. A jet ship propellor consists essentially of a pump which takes in water from the sea and ejects it from nozzles toward the rear (to propel the ship forward). Required the propulsive force developed by such a propellor.

Solution: Let M = weight of water so ejected per unit time, v = velocity of the ship, and V = velocity of the ejected water relative to the ship. The absolute velocity of the jet (relative to the sea) $= V - v$. Hence the amount of momentum produced by the pumping plant (pump, pipes, etc.) per unit time $= M(V - v)$. The direction of this is horizontal and backward; hence the plant exerts a force on the body of water within the passages at any instant equal to $M(V - v)$; the water exerts an equal force forward on the passages, and this is the propulsive force.

223. Conservation of Linear Momentum. — If the algebraic sum of the components — along any line — of the external forces acting on a body equals zero, then the rate of change of the component momentum (along that line) equals zero; hence, if the sum remains zero for any interval of time, the component momentum remains constant. This is known as the principle of *conservation of linear momentum.* It follows that if there are no external forces acting on the body, its linear momentum remains constant. The grand illustration of this principle is furnished by the solar system. Even the nearest stars exert no appreciable attractions on the solar system, and so the members of the system move under the action of their mutual attractions only. Accordingly, the component of the momentum of the system along any line does not change; the linear mo-

mentum is, therefore, constant in amount and direction. It follows that the mass-center of the system moves uniformly, and in a straight line.

224. (Linear) Impulse. — If the magnitude and direction of a force are constant for any interval of time, then the product of the magnitude of the force and the interval is called the impulse of the force for that interval. If the magnitude of the force varies, then the impulse for any interval equals the sum of the elementary impulses Fdt for all the elementary periods of time which make up the interval. If the direction of the force varies, we regard the impulse for any elementary portion of time as a vector quantity having the direction of the force, and then in principle we add (vectorially) the elementary impulses for all the portions of time which make up the interval. That is to say, we integrate Fdt vectorially, arriving at a definite vector quantity.

Units of impulse depend on the units of force and time used.[1] Each unit is named by the names of the units of force and time involved in it. Thus, in the C.G.S. system the unit of impulse is the dyne-second; in the "engineers' system" the unit of impulse is the pound (force) -second.

225. Component of Impulse. — It is evident that the x component of the elementary impulse of a force F is equal to the elementary impulse of its x component; that is $(Fdt)_x = F_x \cdot dt$. Hence, a like relation holds for finite impulses, that is

$$\int (Fdt)_x = \int (F_x dt).$$

For any system of forces, F', F'', etc.

$$\int (F'dt)_x + \int (F''dt)_x + \cdots = \int (F'_x dt) + \int (F''_x dt) + \cdots\cdot =$$
$$\int (F'_x + F''_x + \cdots)dt.$$

Any of these sums we shall call the x component of the impulse of the system of forces.

226. Principle of Impulse and Momentum. — *The component along any line of the impulse of the external forces acting on any body for any interval is equal to the increment for that interval in the component of the momentum of the body along that line.* For, according to Art. 222,

$$F'_x + F''_x + \cdots = \frac{d}{dt}(M\bar{v}_x); \text{ hence } (F'_x + F''_x + \cdots)dt = d(M\bar{v}_x),$$

and
$$\int (F'_x + F''_x + \cdots\cdot)dt = \Delta(M\bar{v}_x).$$

227. Blow. — *Momentum* of a blow, *energy* of a blow, and especially *force* of a blow are terms generally used more or less vaguely. But when one of the two colliding bodies is fixed, then the first two terms are taken

[1] For dimensions of a unit linear impulse, see Appendix A.

to mean the momentum and the kinetic energy respectively of the moving body just before the impact, perfectly definite quantities. If the motion is one of translation, these are Mv and $\frac{1}{2} Mv^2$ respectively.

Force of a blow means the pressure which either of two colliding bodies exerts upon the other. The magnitude of this pressure varies during the collision. Analysis of this variation is beyond the scope of this book, but we discuss the average value of the force of a blow. In the first place, it should be noted that there are two average values of the force of a given blow, a *space-average* and a *time-average*. We explain the distinction by means of a simple example, a protracted blow, the force of which we shall *assume* to vary uniformly with respect to the time; then we shall ascertain how it varies with respect to space and finally compare the two averages. Figure 396 represents the assumed variation; in 20 seconds the force reaches 40 pounds so that $F = 2\,t$ where F denotes the force at any time t. Suppose that this force is exerted between two bodies A and B and so that the force on B say puts B into motion of translation, starting

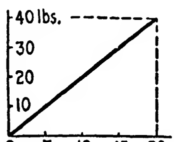


FIG. 396

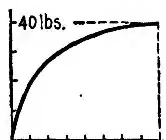


FIG. 397

from rest. Let M denote the mass of B; a, v and s its acceleration, velocity and distance covered at any time t. Then

$$a = \frac{F}{M} = \frac{2\,t}{M}; \quad v = \int a dt = \frac{t^2}{M}; \text{ and } s = \int v dt = \tfrac{1}{3}\frac{t^3}{M}.$$

From the last equation and $F = 2\,t$, it follows that

$$F = 2\,(3\,Ms)^{\frac{1}{3}}.$$

Figure 397 is the graph of this equation; its horizontal length or extent is the total distance covered by B in 20 seconds, $8000 \div 3\,M$.

It is plain from Fig. 396 that the time-average value of F is 20 pounds, and from Fig. 397 that the space-average is much more than 20. It is easy to show that the latter average is 30 pounds.

It will be observed that the space-average is that constant force whose work equals the work done by the (real) varying force (see Art. 194). Likewise the time-average is that constant force whose impulse equals the impulse of the (real) varying force. Hence the space-average equals the quotient of the work done by the force (equal to the kinetic energy produced by the force) and the distance through which the force acted; and the time-average equals the quotient of the impulse of the force (equal to the momentum produced by the force) and the duration of the impulse.

Crushers or crusher gages are small cylinders of copper or lead — one inch diameter and one inch high are common proportions — used in a way described presently to determine the energy and force of a blow. Figure 398 shows several energy-compression curves for $1\frac{1}{2}$- by $1\frac{1}{2}$-inch lead cylin-

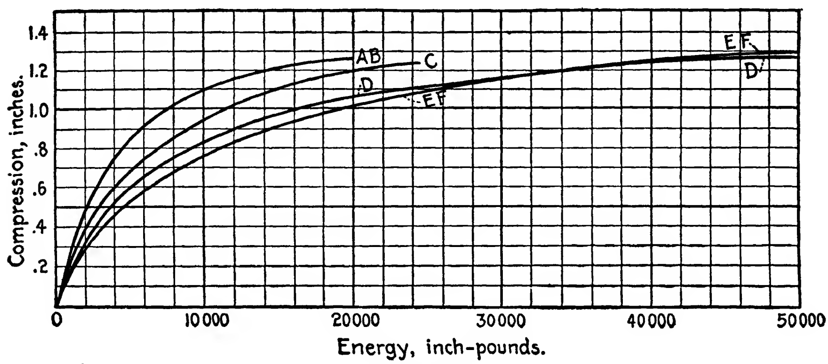


FIG. 398

ders.[1] Curve *A-B* is a so-called static curve, and was obtained by crushing a lead cylinder in an ordinary testing machine at 0.05 inches per minute. The amount of compressing force and the amount of the compression were observed at frequent stages during the test, and from these observations the amount of work done on the cylinder up to each stage was computed. Amounts of compression and corresponding amounts of work were plotted to determine the curve. Curve *C* is a static curve but for a higher speed. *D* is a so-called dynamic curve. It was obtained from drop or impact tests in which each crusher was subjected to a blow from a "hammer" dropped upon it. The hammer weighed 1330 pounds, and the maximum drop used was 38 inches. For each test the amount of compression c was observed and the amount of work 1330 $(h + c)$ was computed, where h = height of drop to the cylinder; and this compression and work were plotted for one point on the curve. Curve *E-F* is a dynamic curve obtained from tests in which the hammers were lighter and the drops higher than for *D*.

Crushers are used as follows to determine the energy of a blow, as of a steam hammer for example. The crusher is placed on the anvil and subjected to the blow. Then the amount of the compression of the crusher is measured, and the corresponding energy is read off from the appropriate compression-energy curve (previously determined from tests on crushers like the one used). The space-average force of such a blow equals the quotient of the energy of the blow and the compression unless c/h is not a small fraction; in that case the space-average $= W(h + c) \div c$. Crushers are also used to determine the powder pressure in guns. The crusher is

[1] *American Machinist*, Vol. 33, Part 1, p. 436 (1910).

held in a steel housing which is placed in the chamber of the gun; the powder pressure is transmitted to the crusher by a steel piston.

228. Collision. — We discuss the changes of motion of one or both colliding bodies due to the collision in certain comparatively simple cases. In most cases of collision the pressures which the colliding bodies exert on each other are enormous compared with other forces acting on the bodies. For example, the space-average pressure between two billiard balls colliding with velocity of 8 feet per second is about 1300 pounds. Therefore in discussing changes of motion of the bodies during collision one may neglect the other (ordinary) forces acting on the bodies, gravity for example; that is, the two bodies jointly are regarded as under the action of no external forces. Hence, according to the principle of conservation (Art. 223), the momentum of the two bodies jointly is not changed by the impact.

If the centers of gravity of two bodies about to collide are moving along the same straight line, then the collision or impact is called *direct;* if otherwise, *oblique.* If the pressures which two colliding bodies exert upon each other during impact are directed along the line joining their centers of gravity, then the impact is called *central;* if otherwise, *eccentric.* These are the kinds of impact called simple, above.

Direct Central Impact. — We assume that the bodies have motions of translation before impact. Since the impact is supposed to be central, the pressure (of impact) on each body acts through the center of gravity of that body and does not turn it. Hence the motion of each body after collision is one of translation. Let A and B be the two bodies,

M_1 and M_2 = their masses,
u_1 and u_2 = their velocities just before impact,
and v_1 and v_2 = their velocities just after impact respectively.

We regard these velocities as having sign; velocity in one direction (along the line of motion) being positive, and that in the other being negative. Then the momentum of the two bodies before impact is $M_1u_1 + M_2u_2$, and after impact it is $M_1v_1 + M_2v_2$. Since the momentums before and after impact are equal,

$$M_1v_1 + M_2v_2 = M_1u_1 + M_2u_2. \qquad (1)$$

The foregoing expressions are correct whether A and B are moving in the same or opposite directions before or after the impact. Thus, if both are moving toward the right before impact, at 8 and 10 feet per second say, their momentum is $8\,M_1 + 10\,M_2$; but if A is moving toward the right and B toward the left, their momentum is $8\,M_1 - 10\,M_2$.

It has been learned experimentally that when two spheres A and B collide directly and centrally the velocity of separation is always less than and opposite to the velocity of approach, and the ratio of these two velocities seems to depend only on the material of the two spheres. The

ratio of the velocity of separation to that of approach (signs disregarded) is called *coefficient of restitution;* it is generally denoted by e. The following are approximate values of e for a few materials,

glass $\frac{15}{16}$, ivory $\frac{8}{9}$, steel and cork $\frac{5}{9}$, wood about $\frac{1}{2}$, clay and putty 0.

Now the velocity of approach equals $u_1 - u_2$ (or $u_2 - u_1$), — the first with reference to A (regarded as fixed) and the second with reference to B (regarded as fixed) —, and the velocity of separation is $v_1 - v_2$ (or $v_2 - v_1$). Since these velocities are opposite in direction, we have

$$-(v_1 - v_2)/(u_1 - u_2) = e, \quad \text{or} \quad -(v_1 - v_2) = e\,(u_1 - u_2) \;.\;.\;. \quad (2)$$

Equations (1) and (2) solved simultaneously for the final velocities v_1 and v_2 give

$$v_1 = u_1 - (1+e)\frac{M_2}{M_1+M_2}(u_1 - u_2);\; v_2 = u_2 - (1+e)\frac{M_1}{M_1+M_2}(u_2 - u_1). \quad (3)$$

If one of the colliding bodies, say B, is fixed, then $u_2 = 0$, and M_2 is the mass of B and its supports, infinitely great. Thus we have $v_1 = -eu_1$.

Oblique Central Impact. — We assume as before that the bodies A and B have a motion of translation before impact; then the pressure on each during the impact acts through the center of gravity and produces no turning. Let U_1 and U_2 = the velocities of A and B before impact; V_1 and V_2 their velocities after impact; u_1 and u_2 = the components of U_1 and U_2 along the line of impact pressure (joining the centers of gravity of A and B when in contact); v_1 and v_2 = the components of V_1 and V_2 along that line; and w_1 and w_2 = the components of U_1 and U_2 at right angles to that line. See Fig. 399 which represents one of several possible ways of oblique collision. Since the impact pressure on either body has no component transversely to the line of pressure XX, the component of the momentum of either body at right angles to XX is not changed. Hence the transverse component of the velocity of either body is not changed by the impact.

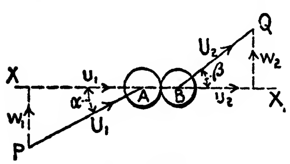


FIG. 399

The longitudinal components are changed as in direct impact, and v_1 and v_2 are given by equations (3). The final velocities V_1 and V_2, therefore, are determined, V_1 by its components v_1 and w_1, and V_2 by its components v_2 and w_2.

Loss of Energy in Impact. — Let L = the loss of kinetic energy; then

$$L = (\tfrac{1}{2}\,M_1U_1^2 + \tfrac{1}{2}\,M_2U_2^2) - (\tfrac{1}{2}\,M_1V_1^2 + \tfrac{1}{2}\,M_2V_2^2).$$

Now $U_1^2 = u_1^2 + w_1^2$, $U_2^2 = u_2^2 + w_2^2$, $V_1^2 = v_1^2 + w_1^2$, and $V_2^2 = v_2^2 + w_2^2$; hence

$$L = \tfrac{1}{2}\,M_1(u_1^2 - v_1^2) + \tfrac{1}{2}\,M_2(u_2^2 - v_2^2).$$

Substituting for v_1 and v_2 their values from equation (3) and simplifying we get

$$L = \frac{1}{2}(1 - e^2)\frac{M_1M_2}{M_1 + M_2}(u_1 - u_2)^2.$$

For perfectly elastic bodies ($e = 1$), $L = 0$. For other bodies $(1 - e^2)$ is not zero but a positive quantity; and since $(u_1 - u_2)$ is not zero, L is always a finite positive quantity. That is, in every collision of bodies not perfectly elastic there is loss of kinetic energy. If the bodies are without elasticity ($e = 0$), the loss $= \frac{1}{2}[(M_1M_2)/(M_1 + M_2)](u_1 - u_2)^2$

The foregoing is essentially Newton's analysis of impact. Several more recent analyses have been made independent of any coefficient of restitution but taking into account the vibrations set up in the colliding bodies. On account of the difficulties of the problem they include only impact of spheres and cylinders end on. Explanation of these analyses falls beyond the scope of this book.[1]

§ 2. Angular Momentum and Impulse

229. Angular Momentum. — The linear momentum of a moving particle is a vector quantity, as explained in Art. 220; the magnitude of the momentum is mv (where m = mass of the particle and v = its velocity), and the direction is that of the velocity. We go further now and assign position to the momentum and to the momentum-vector. The position, or position-line, of the momentum of a moving particle is the line through the particle in the direction of the velocity. Thus the linear momentum of a particle is a "localized" vector quantity, — like a concentrated force, which has magnitude, direction and a definite position, or line of action as it is more commonly called.

By *angular momentum of a particle* about a line is meant the moment of its (linear) momentum about that line. This angular momentum is computed just like the moment of a force about a line. That is, one resolves the momentum into two components, one parallel and one perpendicular to the line, and then takes the product of the perpendicular component and the distance from it to the line. Or, one may resolve the momentum into three components, one being parallel to the line, and then sum the moments of the components.

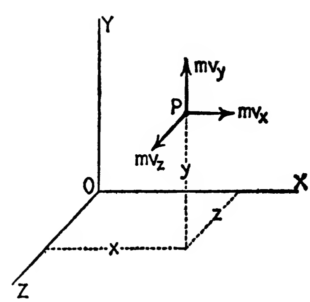


FIG. 400

[1] See Love's *Theory of Elasticity*, Vol. 2; *Nature*, Vol. 88, p. 531 (1912) for an instructive paper by Prof. Hopkinson, on "The Pressure of a Blow"; also *Journal of the Franklin Institute*, Vol. 172, p. 22 (1911) for an account of some determinations of the time of impact of metal spheres.

Thus let mv be the momentum of a particle P (Fig. 400) of a body not shown; the three components of mv are shown. About the x-axis, say, mv_x has no moment, and the angular momentum is

$$mv_z y - mv_y z.$$

By *angular momentum of a body* about a line is meant the sum of the angular momentums of all the particles of the body about that line. For the axes of a coördinate frame $OXYZ$,

$$\begin{aligned} h_x &= \Sigma(mv_z y - mv_y z), \\ h_y &= \Sigma(mv_x z - mv_z x), \\ h_z &= \Sigma(mv_y x - mv_x y), \end{aligned}$$

where h_x denotes angular momentum about the x-axis, and similarly h_y and h_z.

In the case of a rigid body rotating about a fixed axis the formula for the angular momentum about that axis reduces readily to $h = \omega I$, where I = the moment of inertia of the body about the axis of rotation. This expression can also be derived directly as follows: Let m_1, m_2, etc., = the masses of the particles of the body; r_1, r_2, etc., = the distances of the particles respectively from the axis of rotation; and ω = the angular velocity of the body. Then the linear velocities of the particles are respectively $r_1\omega$, $r_2\omega$, etc. (Art. 138), and their linear momentums are $m_1r_1\omega$, $m_2r_2\omega$, etc. These momentums are perpendicular to the axis of moments; hence the angular momentums are $m_1r_1\omega r_1$, $m_2r_2\omega r_2$, etc. And since these are of the same sign, the angular momentum of the body is $m_1r_1^2\omega + m_2r_2^2\omega + \cdots = \omega\Sigma mr^2 = \omega I$.

The *unit of angular momentum*[1] is the angular momentum of a particle having unit (linear) momentum and an arm about the line in question equal to unit length. Any unit of angular momentum is named by naming the units of linear momentum and length used. Thus in the C.G.S. system the unit is the dyne-second-centimeter; in the Foot-Pound (force) -Second System (see page 194), the pound (force) -second-foot.

230. Rate of Change of Angular Momentum About a Line. — Differentiation of say the formula for h_x of the preceding article gives

$$\frac{dh_x}{dt} = \Sigma\left[m\left(\frac{dv_z}{dt}y + v_z\frac{dy}{dt}\right) - m\left(\frac{dv_y}{dt}z + v_y\frac{dz}{dt}\right)\right].$$

Since $$\frac{dy}{dt} = v_y, \quad \frac{dz}{dt} = v_z, \quad \frac{dv_y}{dt} = a_y \quad \text{and} \quad \frac{dv_z}{dt} = a_z,$$

$$\frac{dh_x}{dt} = \Sigma(ma_z y - ma_y z).$$

Similarly $$\frac{dh_y}{dt} = \Sigma(ma_x z - ma_z x), \quad \text{and} \quad \frac{dh_z}{dt} = \Sigma(ma_y x - ma_x y).$$

[1] For dimensions of unit angular momentum, see Appendix A.

231. Principle of Torque and Angular Momentum. — *The torque or moment of the system of external forces acting on any body about any line equals the rate at which the angular momentum of the body about that line is changing.* Thus,

$$T_x = \frac{dh_x}{dt}; \; T_y = \frac{dh_y}{dt} \; \text{ and } \; T_z = \frac{dh_z}{dt}.$$

We prove the first equation, where the x-axis is *any* line. Let P, Fig. 401, be any particle of the body, not shown; a_x, a_y and a_z the axial components of the acceleration of P; x, y and z the (varying) coördinates of P, and m = its mass. Then the resultant of all forces acting on P is ma, and its axial components are as marked in the figure. The torque about OX of all forces acting on P is $ma_zy - ma_yz$, and

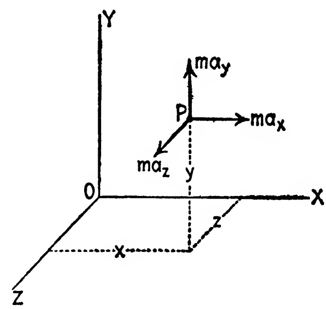


FIG. 401

$$\left.\begin{array}{r}\text{the torque of all forces acting}\\ \text{on all the particles of the body}\end{array}\right\} = \Sigma(ma_zy - ma_yz).$$

Now the left-hand member of this equation includes the moments of all forces, external and internal, that act on the body. Since the internal forces occur in pairs of equal, opposite and colinear forces, *their* moments cancel; that is *their* (total) torque is zero, and the value of the left-hand member is the same as that of the torque of the external forces, that is T_x.

The right-hand member is dh_x/dt (see preceding article), that is the rate at which the angular momentum about the x-axis is changing. Thus the first of the equations above has been proved.

EXAMPLE. Figure 402 represents, in principle, a hydraulic motor sometimes called "Barker's Mill." Essentially, the motor consists of a horizontal cylinder AB, mounted on a vertical pivot C, and an inlet D connected by a water-tight sleeve joint to a feed pipe E. On opposite sides of the cylinder and near its ends there are orifices or nozzles through which the water escapes horizontally. The water turns the motor in the opposite direction. Required the torque exerted by the water on the motor.

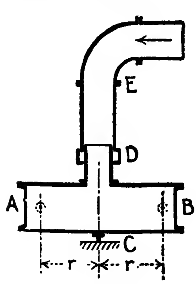


FIG. 402

Solution: Let M = the mass of water escaping per unit time, v = the velocity of escape relative to the orifices, and ω = the angular velocity of the motor. The mass of water which escapes in a short interval of time Δt is $M\Delta t$; and, since the absolute velocity of escape = $v - r\omega$ (Art. 155), the angular momentum of this water about the axis of rotation is $(M\Delta t)$ $(v - r\omega)r$. Hence the rate at which the motor gives angular momentum to the water is

$$M(v - r\omega)r,$$

and this equals the torque of the motor on the water; also the torque of the water on the motor.

232. Conservation of Angular Momentum. — If the torque (about any line) of the external forces acting on a body equals zero, then the rate of change of the angular momentum of the body about that line equals zero; hence, if the torque remains zero for any interval of time, then the angular momentum remains constant. This is known as the principle of the *conservation of angular momentum.* It can be well illustrated by means of the apparatus on which the man (Fig. 403) is standing. It consists of a metal plate A supported on balls in suitable circular races in A and B so that A can be rotated about the line C with very little friction resistance; B is fixed. Imagine that a man has mounted the plate A and holds a balancing pole as shown, all being at rest; then the angular momentum of the man-plate-pole system about CC equals zero. Now suppose that the man exerts himself in any way, to move the pole about for example, but touches nothing except A and the pole. The only external forces acting on the system are gravity, the reactions of the balls on A, and the air pressure. The first has no torque about C; the other two have very little and are negligible here. Hence there is no external torque about C, and the angular momentum of the system about C equals zero always. This is strikingly illustrated if the man, without moving his feet on the plate, trys to rotate the pole (over his head as shown) about C. In doing so, he and A begin to rotate in the opposite direction. If I and I' = the moments of inertia of man (and A) and the pole respectively about C, and ω and ω' = their angular velocities at any instant, then the principle requires that the angular momentums $I\omega$ and $I'\omega'$ shall be equal (and opposite). Or, imagine that the man-plate-pole system is given an angular velocity by external means (the man holding the rod as shown, say), and then left to itself. If now the man should change the pole into a vertical position before him, he would reduce the moment of inertia of the system (about C) very materially; and since the angular momentum must remain constant, the angular velocity of the system would increase accordingly.

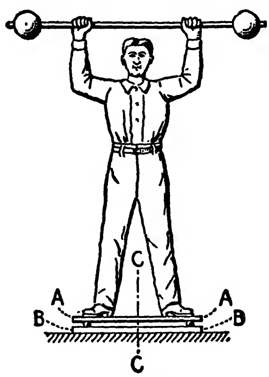


FIG. 403

The grand illustration of the principle of conservation of angular momentum is furnished by the solar system. The system moves under the influence of no external forces; hence the angular momentum of the system about any line remains constant. The angular momentum about a certain line through the mass-center of the system is greater than that about any other such line. The line is known as the invariable axis of the system — a plane perpendicular to it as the invariable plane — and "is the nearest approach to an absolutely fixed direction yet known."

233. Angular Impulse. — If the line of action of a force is fixed in position then the "angular impulse of that force for any interval about any line" is the moment of the impulse-of-the-force-for-the-interval about that line. The moment of an impulse is computed just like moment of a force or angular momentum about a line; that is, one resolves the impulse into two components, one parallel and one perpendicular to the line and then takes the product of the perpendicular component and the distance from it to the line. Or, one may resolve the impulse into three rectangular components, one being parallel to the line, and then sum the moments of these three components.

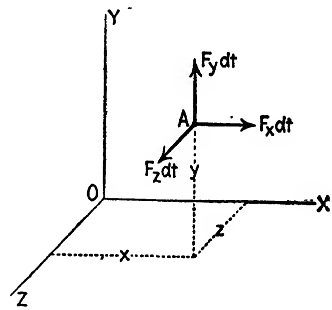


FIG. 404

Thus let F (Fig. 404) be a force acting at a point A of a body not shown. The x, y and z components of the elementary impulse Fdt are as marked. The angular impulse of the force for the elementary time dt about the x-axis, say, is

$$(F_z dt)y - (F_y dt)z = (F_z y - F_y z)dt,$$

and the angular impulse of the force for any finite interval of time is

$$\int (F_z y - F_y z)dt.$$

By angular impulse about any line of a system of forces is meant the sum of the angular impulses of the forces about that line. The formula is

$$\int (F'_z y' - F'_y z')dt + \int (F''_z y'' - F''_y z'')dt + \cdots\cdots,$$

$$\text{or} \int \Sigma(F_z y - F_y z)\, dt, \qquad \text{or} \int T_x dt,$$

where T_x denotes the torque of the system of forces about the x-axis.

Units of angular impulse depend on the units of force, time, and length used. Each unit is named by giving the name of the units of force, time, and length used. Thus, in the C.G.S. system, the unit of angular impulse is the dyne-second-centimeter; in the "engineers' system," the pound-force-second-foot.

234. Principle of Angular Impulse and Momentum. — *The* (total) *angular impulse about any line for the external forces acting on any body for any interval of time equals the increment in the angular momentum of the body about that line.* This follows easily from Art. 231, where it is shown that

$$T_x = \frac{dh_x}{dt}, \quad \text{or} \quad T_x dt = dh_x\,; \quad \text{hence} \int T_x dt = \Delta h_x.$$

EXAMPLE. The *ballistic pendulum* is a device for determining the velocity of a bullet; it consists essentially of a pendulum having as a bob a heavy block, box of sand, or

similar object, into which the bullet is fired. The impact of the bullet causes the pendulum to swing upward and from the amount of this swing the velocity of the bullet can be ascertained. It is required to develop a formula for velocity of bullet in terms of quantities easily measured in a test.

Solution: Let M denote mass of pendulum, k its radius of gyration about its axis of oscillation, c distance from the axis to its mass-center; r distance from the axis to path of striking bullet, m mass of bullet; v striking velocity, ω maximum angular velocity of the pendulum, and θ angle through which pendulum rises on account of the impact. Consider the shot and pendulum as a "system." Just before and just after the impact the angular momentums of the system about the axis are respectively

$$mvr \quad \text{and} \quad (mr\omega)r + Mk^2\omega.$$

During impact the only external forces acting on the system are gravity and the reaction of the support of the pendulum. These forces have no angular impulses about the axis during the impact; hence, there is no change in the angular momentum of the system during that period, that is the two expressions above are equal; hence,

$$v = \frac{(mr^2 + Mk^2)\omega}{mr}.$$

At the conclusion of the impact period the kinetic energy of the system is

$$\tfrac{1}{2}\, m(r\omega)^2 + \tfrac{1}{2}\, Mk^2\omega^2.$$

This is "expended" in "overcoming" gravity during the rise of the pendulum, that is,

$$\tfrac{1}{2}\,(mr^2 + Mk^2)\omega^2 = (mgr + Mgc)\,(1 - \cos\theta).$$

Substitution of the value of ω given by this equation in the formula for v above gives

$$v^2 = \frac{(mr^2 + Mk^2)\,(mr + Mc)}{m^2r^2}\, 2\,g(1 - \cos\theta).$$

CHAPTER XIV

THREE DIMENSIONAL MOTION OF A RIGID BODY

235. Introductory Remark. — The kinematics of three dimensional motion of a rigid body is discussed in Art. 145 to 151; for the kinetics there are available the principle of the motion of mass-center (Art. 173), and the principle of torque and angular momentum (Art. 231). They are restated here in equation form for convenience:

$$\Sigma F_x = M\bar{a}_x, \qquad \Sigma F_y = M\bar{a}_y, \qquad \Sigma F_z = M\bar{a}_z \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$T_x = \frac{dh_x}{dt}, \qquad T_y = \frac{dh_y}{dt}, \qquad T_z = \frac{dh_z}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We shall use also "angular momentum about a point" which we explain next. It is helpful in getting values of the rates dh_x/dt, etc.

§ 1. Angular Momentum About a Point

236. Definitions. — By angular momentum of a *particle* about a point is meant the moment about that point of its momentum. Thus if m and v are the mass and velocity of a particle P (Fig. 405) and p is the perpendicular distance from the point to the line drawn through the particle and in the direction of its motion at the instant in question, then the angular momentum, about the point, of the particle at the instant is mvp. This angular momentum is represented by a vector which we call r, through the point O, perpendicular to the plane of O and the momentum vector mv; the length of r is made equal (according to some convenient scale) to mvp, and the arrow on the vector is fixed in accordance with the usual rule, namely, the arrow agrees with the advance of a right-hand screw turning in a fixed nut at the fixed point in the direction suggested by the momentum vector.

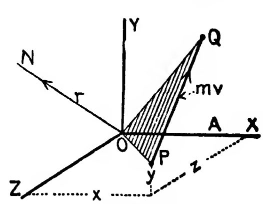


FIG. 405

By angular momentum of a *body* about a fixed point is meant the resultant of the angular momentums about that point of all its particles. Thus the angular momentum of the body about a fixed point is represented by a definite vector R, namely, the resultant of the vectors which represent the angular momentums, about the point, of the particles of the body.

237. Relation Between Angular Momentum About a Point and About a Line Through the Point. — Since angular momentum about a point is a vector quantity, it can be resolved into components, for example along

coördinate axes through the point. We now prove *that the component of the angular momentum of a body about a point, along any line through the point, equals the angular momentum of the body about that line.* We prove this proposition for a single particle. Then it follows that the proposition holds for a body. We take the point as an origin of coördinates and the line as x-axis; then we are to prove that

$$r_x = mv_z y - mv_y z.$$

As in Fig. 405 let P (Fig. 406) be the particle; PQ the momentum vector; α, β and γ respectively the direction angles of PQ; p the perpen-

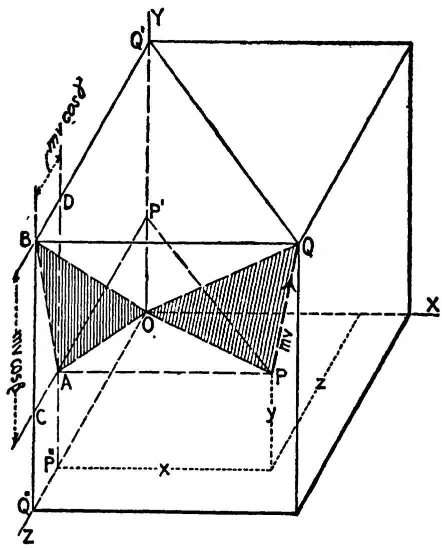


FIG. 406

dicular distance from O to PQ, and r the angular momentum of P about O; also ϕ = angle between the angular momentum vector (not shown) and the x-axis. Now,

$$r = PQ \times p = 2 \times (\text{area } POQ); \text{ and } r_x = (2 \times \text{area } POQ) \cos \phi.$$

Since ϕ is the angle between normals to the planes POQ and YOZ, it is equal to the angle between the planes; hence (area POQ) $\cos \phi$ = projection of POQ on the yz coördinate plane. In the figure, this projection is AOB, A and B being the projections of P and Q respectively on the yz plane. That is,

$$r_x = 2 \times AOB = 2\,(BOQ'' - ABC - AOP'' - AP''Q''C).$$

To calculate these last four areas, we need the lengths of BC ($= P'Q'$) and BD ($= P''Q''$). Obviously, P' and Q' are the projections of P and Q on the y-axis, and P'' and Q'' are the projections of P and Q on the z-axis;

hence $P'Q'$ and $P''Q''$ are the projections of the vector PQ on the y- and z-axes, or

$$P'Q' = mv \cos\beta, \quad \text{and} \quad P''Q'' = mv \cos\gamma.$$

Now
$$BOQ'' = \tfrac{1}{2}(mv\cos\gamma + z)(mv\cos\beta + y),$$
$$ABC = (mv\cos\gamma)\,\tfrac{1}{2}(mv\cos\beta),$$
$$AOP'' = \tfrac{1}{2}zy, \quad \text{and} \quad AP''Q''C = (mv\cos\gamma)y.$$

Substituting and simplifying gives

$$AOB = \tfrac{1}{2}(mv\cos\gamma\cdot y - mv\cos\beta\cdot z) = \tfrac{1}{2}(mv_z y - mv_y z),$$

or
$$r_x = mv_z y - mv_y z.$$

238. Relation Between Rates of Change of Angular Momentum About a Point and About a Line Through the Point. — As the body moves, its angular momentum H about any fixed point continually changes in magnitude and direction. Imagine a vector OD representing H, O being at the fixed point, say; D moves about, and its velocity at any instant represents the rate at which H is changing at that instant. (See Art. 127 for proof of strictly analogous case.) Thus the instantaneous rate of change of angular momentum about a point is a vector quantity; its magnitude is the magnitude of the speed of the point of the vector OD, and its direction is the direction of the motion of the end D at the instant, that is tangent to the path of D (Fig. 407).

Being a vector, the rate of change of angular momentum dH/dt can be resolved into components. We now show that the component of this

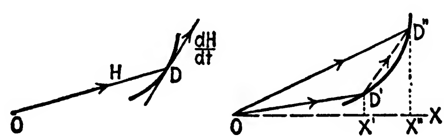


FIG. 407

rate along any fixed line equals the rate at which the angular momentum about that line is changing. That is, if the line is an x-axis, say,

$$(dH/dt)_x = dh_x/dt.$$

Let OD' and OD'' represent the angular momentum H at the beginning and end of any interval Δt; then vector $D'D''$ represents ΔH. If OX is an x-axis then OX' and OX'' represent h_x at the beginning and end of Δt, and $X'X''$ represents Δh_x. Evidently,

the x component of $\Delta H = \Delta h_x$;

hence the x component of $\Delta H/\Delta t = \Delta h_x/\Delta t$,

and the x component of $dH/dt = dh_x/dt$

which was to be proved.

§ 2. Spherical Motion

239. Calculation of Angular Momentum of a Rigid Body in Spherical Motion About a Line Through the Fixed Point. — The following general formulas, numbered (1), for angular momentum about fixed x, y and z axes (see Art. 229) can be transformed, for the case of a rigid body having spherical motion, into equations (2). It will be noticed that (2) involve angular and not, as (1), linear velocities.

$$h_x = \Sigma m(v_z y - v_y z), \quad h_y = \Sigma m(v_x z - v_z x), \quad h_z = \Sigma m(v_y x - v_x y). \quad (1)$$

$$\left.\begin{aligned} h_x &= +I_x\omega_x - K_{yx}\omega_y - K_{zx}\omega_z \\ h_y &= -K_{xy}\omega_x + I_y\omega_y - K_{zy}\omega_z \\ h_z &= -K_{xz}\omega_x - K_{yz}\omega_y + I_z\omega_z \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (2)$$

I_x, I_y and I_z = the moments of inertia of the body about the x, y and z axes respectively; or symbolically,

$$I_x = \Sigma m(y^2 + z^2), \qquad I_y = \Sigma m(z^2 + x^2), \qquad I_z = \Sigma m(x^2 + y^2).$$

K_{yz}, K_{zx} and K_{xy} respectively = the products of inertia of the body with respect to the two coördinate planes intersecting in the x, y and z axes (Appendix B); or symbolically,

$$K_{yz} = \Sigma myz, \qquad K_{zx} = \Sigma mzx, \qquad K_{xy} = \Sigma mxy.$$

Symbols ω_x, ω_y and ω_z denote the axial components of the angular velocity of the body. Equations (2) may be deduced from equations (1) by substituting for v_x, v_y and v_z their values from the equations of Art. 149, and then simplifying.

If the coördinate axes x, y and z are principal axes of the body at the fixed point (Appendix B), then K_{yz}, K_{zx} and $K_{xy} = 0$; and

$$h_x = I_x\omega_x, \qquad h_y = I_y\omega_y, \qquad \text{and} \qquad h_z = I_z\omega_z. \quad . \; . \; . \; . \quad (3)$$

240. Calculation of Angular Momentum About the Fixed Point. — The practical method is not to add vectorially the angular momentums of all the particles of the body but this: calculate the angular momentums of the body about three rectangular axes through the point, then find the vectorial sum of these angular momentums; this sum is the angular momentum about the point.

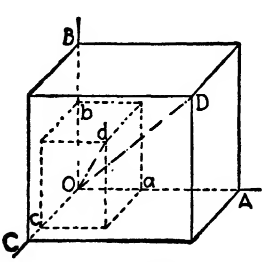


FIG. 408

It is worth noting that the axis of rotation of a body with a fixed point (Art. 147), and the vector of the angular momentum about the point are in general not coincident. Thus let O (Fig. 408) be the fixed point of a body not shown, and OA, OB and OC the principal axes of the body at O. If Oa, Ob and Oc represent the components (ω_1, ω_2 and ω_3) along the axes respectively of the angular velocity ω, then Od represents ω and Od is the axis of rotation. If OA, OB and OD represent the angular momentums h_1, h_2 and h_3 (or

$I_1\omega_1$, $I_2\omega_2$ and $I_3\omega_3$) about the axes respectively, then OD represents the angular momentum H. Obviously Od and OD coincide only

$$\text{if}\quad h_1 : h_2 : h_3 : : \omega_1 : \omega_2 : \omega_3, \quad \text{that is, if} \quad I_1 = I_2 = I_3.$$

EXAMPLE. Suppose that the sphere described in Ex. 1, Art. 148, weighs 100 lb. It is required to determine the angular momentum of the sphere when rolling as described there.

Solution: The principal axes of the sphere at the fixed point O (Fig. 409) are $O1$ and any two lines perpendicular to each other and to $O1$ at O, as $O2$ and $O3$ (not shown). The components of the angular velocity of the sphere along $O1$, $O2$ and $O3$ respectively are

$$\omega_1 = -7.35 \text{ rad/sec.}, \quad \omega_2 = 2.95 \text{ rad/sec.}, \quad \text{and} \quad \omega_3 = 0$$

(see Ex. 1, Art. 148). The mass of the sphere is $100/32.2 = 3.10$ slugs; the moments of inertia of the sphere are as follows:

$$I_1 = \tfrac{2}{5}\, 3.10 \times 2^2 = 4.97 \text{ sl-ft.}^2$$
$$I_2 = I_3 = 4.97 + 3.10 \times 5.01^2 = 82.8 \text{ sl-ft.}^2$$

Hence the angular momentums of the sphere about the principal axes are

$$h_1 = -4.97 \times 7.35 = 36.5, \quad h_2 = 82.8 \times 2.95 = 244,$$
and $$h_3 = 0.$$

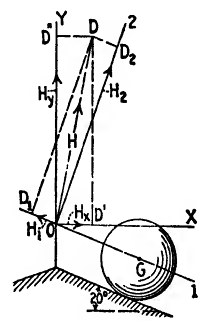


FIG. 409

OD_1 and OD_2 represent the values of h_1 and h_2 respectively; they are laid off from O in directions to agree with the vectors representing the components of the angular velocity. The diagonal OD of the parallelogram represents the angular momentum H of the sphere about O.

$$OD = \sqrt{36.5^2 + 244^2} = 247 \text{ sl-ft}^2\text{/sec.},$$

and angle $D_2OD = \tan^{-1}(36.5 \div 244) = 8° 30'$.

241. Rate of Change of Angular Momentum About a Line Through the Fixed Point. — If the x, y and z axes are fixed, then expressions for the rates of change of h_x, h_y and h_z may be obtained by (ordinary) differentiation of the right-hand members of equation 2 (Art. 239). The results arrived at will not be simple, since in general all I's, K's and ω's vary with the time. Simpler results for these rates may be deduced by means of the following methods.

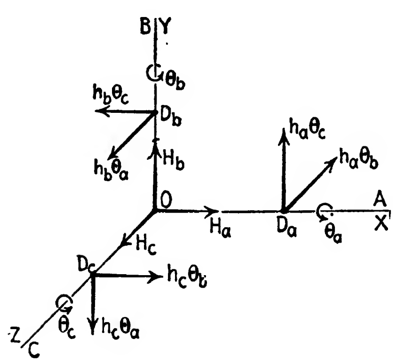


FIG. 410

(i) The simplest method in principle at least is as follows: find the rate of change of the angular momentum about the fixed point as explained in Art. 238, and then resolve this rate along the fixed axes. The three components equal the desired rates.

(ii) Method of moving axes: Let O (Fig. 410) be the fixed point of the moving body (not shown); OA, OB and OC a set of axes moving with the

body or otherwise; θ the (varying) angular velocity of the frame $OABC$, and θ_a, θ_b and θ_c the components of θ along OA, OB and OC respectively. Let OX, OY and OZ be a set of fixed axes with which the moving axes coincide as shown at the instant in question. Then at that instant, $h_a = h_x$, $h_b = h_y$ and $h_c = h_z$. If these moving axes are fixed in the body, then $\theta = \omega$, $\theta_a = \omega_x$, $\theta_b = \omega_y$ and $\theta_c = \omega_z$.

In general all h's vary; the corresponding ones (h_a and h_x, h_b and h_y, h_c and h_z) do not vary at equal rates, even at the instant when the sets of axes coincide. At the instant of coincidence

$$\frac{dh_x}{dt} = \frac{dh_a}{dt} - h_b\theta_c + h_c\theta_b,$$

$$\frac{dh_y}{dt} = \frac{dh_b}{dt} - h_c\theta_a + h_a\theta_c,$$

$$\frac{dh_z}{dt} = \frac{dh_c}{dt} - h_a\theta_b + h_b\theta_a,$$

where the derivatives, the h's and the θ's all pertain to the instant in question.

We give a derivation of the first of the above equations; the other two could be derived in a similar manner: At all times during the motion, even when the two sets of axes (or "frames" $OABC$ and $OXYZ$) are not coincident, *h_x is equal to the algebraic sum of the components along OX of h_a, h_b and h_c;* the increment in h_x is equal to the algebraic sum of the x components of the increments in h_a, h_b and h_c; and the rate of change of h_x (or dh_x/dt) is equal to the algebraic sum of the x components of the rates of change of h_a, h_b and h_c. In general, h_a, h_b and h_c change in both magnitude and direction. We consider separately their contributions to the rate of change of h_x for the instant when the two sets of axes or "frames" $OABC$ and $OXYZ$ are coincident, due to (i) changes in mere magnitudes of h_a, h_b and h_c and (ii) changes in direction of h_a, h_b and h_c.

(i) When the frame $OABC$ is permanently coincident with $OXYZ$, then at all times $h_x = h_a$, $h_y = h_b$ and $h_z = h_c$; changes in magnitude of h_b and h_c do not affect h_x, but change in magnitude of h_a does; and in fact any change in h_a produces an equal change in h_x and the rate of change in h_x would be equal to dh_a/dt.

(ii) When the frame $OABC$ is rotating and h_a, h_b and h_c are constant in magnitude, then at the instant when the frames are coincident the rates of change of the vectors, h_a, h_b and h_c, due to the component rotations θ_a, θ_b and θ_c about OA, OB and OC (or OX, OY and OZ) are as indicated in the figure. Thus, due to rotation about OA, with angular velocity θ_a, the rates of h_a, h_b and h_c respectively are 0, $h_b\theta_a$, $h_c\theta_a$, directed as shown in the figure; due to rotation about OB with angular velocity θ_b, the rates respectively are $h_a\theta_b$, 0 and $h_c\theta_b$, directed as shown; and due to rotation about OC with angular velocity θ_c, they are $h_a\theta_c$, $h_b\theta_c$ and 0, directed as

shown. The figure shows clearly that only two of these six rates have components along the x-axis, and that these components are $-h_b\theta_c$ and $h_c\theta_b$.

Finally, the entire rate of h_x is the algebraic sum of the three constituents, namely, from (i) dh_a/dt and from (ii) $-h_b\theta_c$ and $h_c\theta_b$.

EXAMPLE. The rate of change of the angular momentum of the sphere about O in the preceding example is required; also the rates of change of the angular momentums about the axes OX, OY and OZ (Fig. 409); and about $O1$, $O2$ and $O3$.

Solution: The rate of change of the angular momentum H equals the velocity of the point D. Evidently D moves (as the sphere rolls) in a horizontal circle whose radius is

$$247 \sin (20° - 8° 30') = 49.2 \text{ sl/ft}^2\text{/sec.}$$

And since D makes 30 trips per minute around OY, the velocity of D is

$$2\pi \times 49.2 \times 0.5 = 154.5 \text{ sl/ft}^2\text{/sec}^2.$$

This velocity, for the position of the sphere shown, is downward through the paper; that is, in the negative z-direction. The components of this rate along OX, OY and OZ are respectively

$$dh_x/dt = 0, \quad dh_y/dt = 0, \quad \text{and} \quad dh_z/dt = -154.5.$$

The components of this rate along the principal axes are respectively

$$dh_1/dt = 0, \quad dh_2/dt = 0, \quad \text{and} \quad dh_3/dt = -154.5$$

242. Sense and Sign of Torque and of Rate of Change of Angular Momentum. — The equation $T_x = dh_x/dt$ (Art. 231) requires of course that the signs of the two members be alike. The rule of signs for torque is explained in Art. 18; it is implied for rate of change of angular momentum in Art. 230 in a manner to agree precisely with the rule for torque. In this chapter we have emphasized the vector aspect of rate of change of angular momentum, and it is necessary to note how the sign or sense of T_x ties in with the arrow or sense of the vector representing dh_x/dt.

Let the direction O to X (Fig. 411) be the positive direction along the line OX, and suppose that the vector OD represents dh_x/dt for a body not shown and for a particular instant; then dh_x/dt is positive. Also T_x is positive which means that the sense or rotation of the torque is the same as the direction in which one must turn a right-hand screw in a nut fixed on the line OX so as to make the screw advance in the *positive* direction. If dh_x/dt is positive, then positive h_x is being taken on by the body to produce which requires positive T_x; conversely if T_x is positive then positive h_x is being taken on by the body.

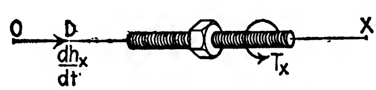


FIG. 411

EXAMPLE 1. G (Fig. 412) is a sphere; OG is a rod fixed to the sphere, and OY is a shaft resting in a deep socket in the floor. The rod extends loosely through a hole in

the shaft and has a nut on its end as shown; the shaft can turn easily in the socket. The sphere weighs 100 lbs. and is 4 ft. in diameter. It is made to roll and travel around the shaft 30 times per minute by means of a horizontal force perpendicular to the rod applied at a point midway between O and G. It is required to determine the reaction of the floor on the sphere, the reaction of the shaft on the rod, and the applied force.

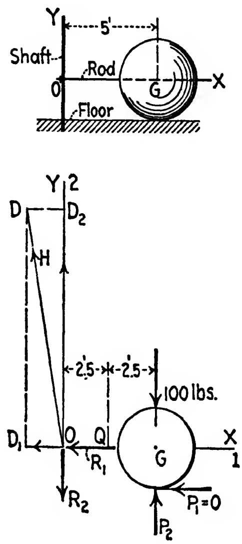


FIG. 412

Solution: Figure 412 represents the sphere and rod as one body with all external forces so far as they can be shown in the plane of the paper. R_1, R_2 and R_3 denote axial components of the reaction of the shaft on the rod, P_1, P_2 and P_3 axial components of the reaction of the floor on the sphere; and Q the applied force. R_3 and P_3 are perpendicular to the paper; the component P_1 is obviously zero. First we calculate quantities for the right-hand members of Eq. 1 and 2, Art. 235: The mass-center of the sphere describes a (horizontal) circle with constant speed $2\,\pi\,5 \times \frac{1}{2} = 15.7$ ft/sec. Hence the acceleration of the mass-center is $15.7^2 \div 5 = 49.2$ ft/sec/sec., directed from G to O.

The angular velocity of the shaft is 30 rev/min.; the angular velocity of the sphere relative to the rotating shaft is $(2\,\pi\,5 \div 2\,\pi\,2) \times 30 = 75$ rev/min. Hence

$$\omega_1 = 75 \times 2\,\pi/60 = 7.85 \text{ rad/sec.}$$
$$\omega_2 = 30 \times 2\,\pi/60 = 3.14 \text{ rad/sec.}$$

The mass of the sphere is $100/32.2 = 3.10$ slugs. Hence (see Appendix B),

$$I_1 = \tfrac{2}{5}\,3.10 \times 2^2 = 4.96 \text{ sl-ft.}^2;$$

and

$$h_1 = 4.96 \times 7.85 = 39.0 \text{ sl-ft}^2\text{/sec.}$$

Supposing that the travel of the sphere around the shaft is counterclockwise when viewed from above, then OD_1 and OD_2 may represent H_1 and H_2 respectively, and OD represents H. The rate at which H changes is the velocity of D (Art. 238); or

$$39.0 \times 3.14 = 122 \text{ sl-ft}^2\text{/sec.}^2,$$

and it is directed outward from the paper. Hence,

$$\frac{dh_x}{dt} = 0, \quad \frac{dh_y}{dt} = 0, \quad \text{and} \quad \frac{dh_z}{dt} = +122.$$

Substitution in Eq. (1) and (2) of Art. 235 gives

$$-R_1 = -3.10 \times 49.2, \qquad P_3 \times 2 = 0,$$
$$P_2 - R_2 - 100 = 0, \qquad Q \times 2.5 - P_3 \times 5 = 0,$$
$$R_3 + P_3 - Q = 0, \qquad P_2 \times 5 - 100 \times 5 = 122.$$

Simultaneous solution of the equations gives

$$Q = 0, \qquad P_2 = 124, \qquad P_3 = 0$$
$$R_1 = 154, \qquad R_2 = 24, \qquad R_3 = 0.$$

EXAMPLE 2. Referring to the example of Art. 240, suppose that the sphere and rod are driven around the shaft (counterclockwise when viewed from above) by means of a horizontal force applied perpendicularly to the rod at a point midway between O and G. It is required to determine the reaction R of the shaft on the rod at O and the reaction P of the cone on the sphere, neglecting rolling resistance and the mass of the rod.

Solution: Figure 413 is the free body diagram. The external forces acting on the sphere-rod body are gravity 100 lb., the applied force Q, R, and P. For convenience R is replaced by its components parallel to axes x, y and z, but the third component does not show. Also P is replaced by its components parallel to axes $O1$, $O2$ and $O3$, the third not showing. Evidently $P_1 = 0$.

The mass-center G moves in a horizontal circle with constant speed; hence the acceleration $\bar{a}$ of the mass-center is as represented and its magnitude is $4.68 \times 3.14^2 = 46.2$ ft/sec/sec. The mass of the sphere is $100/32.2 = 3.105$ slugs. It is shown in the solution of the example in Art. 241 that

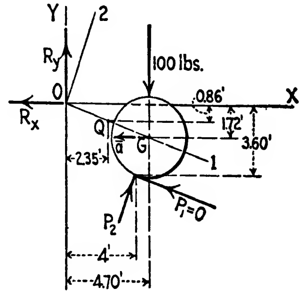


FIG. 413

$$dh_x/dt = 0, \quad dh_y/dt = 0, \quad \text{and} \quad dh_z/dt = -154.5.$$

Substitution in Eq. (1) and (2) of Art. 235 gives

$$-R_x + P_2 \sin 20^\circ = -3.105 \times 46.2, \qquad Q \times 0.86 - P_3 \times 3.60 = 0,$$
$$R_y - 100 + P_2 \cos 20^\circ = 0, \qquad Q \times 2.35 - R_3 \times 4 = 0,$$
$$R_z - Q + P_3 = 0, \qquad P_2 \times 5.01 - 100 \times 4.70 = -154.5.$$

Simultaneous solution of these equations gives

$$Q = 0, \qquad P_2 = 63.0, \qquad P_3 = 0,$$
$$R_x = 168.1, \qquad R_y = 40.8, \qquad R_z = 0.$$

§ 3. Any Three-dimensional Motion of a Rigid Body

243. Angular Momentum About a Line Through the Mass-center and Fixed in Direction. — Let O' (Fig. 414) be the mass-center of any moving body not shown; $OXYZ$ a set of fixed axes; $O'X'Y'Z'$ a second set of axes always parallel to the first set; P any particle of the body; (x, y, z) and (x', y', z') respectively the coördinates of P with respect to the two sets of axes; v and v' the velocities of P with respect to the sets; and m = mass of P. Also $(x, \bar{y}, \bar{z})$ denote the coördinates and $\bar{v}$ the velocity of the mass-center with respect to the fixed set of axes. Then the angular momentum of P about OX say is

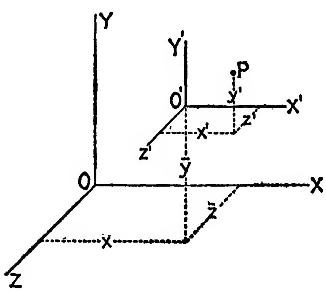


FIG. 414

$$(mv_zy - mv_yz)$$

and the angular momentum of the entire body about OX is

$$h_x = \Sigma(mv_zy - mv_yz). \quad \ldots \ldots \quad (1)$$

Now $\quad v_y = \bar{v}_y + v'_y$, and $v_z = \bar{v}_z + v'_z$; also $y = \bar{y} + y'$ and $z = \bar{z} + z'$.

Hence, $\quad h_x = \Sigma[m(\bar{v}_z + v'_z)(\bar{y} + y') - m(\bar{v}_y + v'_y)(\bar{z} + z')]. \quad \ldots \quad (2)$

Now let us see what this becomes if O is taken at the place where the

mass-center happens to be at any particular instant. Then $\bar{y} = \bar{z} = 0$, and $\Sigma my' = \Sigma mz' = 0$;

$$h_x = \Sigma(mv'_z y' - mv'_y z'). \quad \ldots \ldots \ldots \ldots \quad (3)$$

It should be noticed particularly that h_x does not depend at all on the motion of the mass-center; that is, one may calculate the angular momentum of a body about a line through the mass-center just as though that point were fixed. For instance the angular momentums of a body about the principal axes through its mass-center, even if that point is moving, are given by Equation 3 of Art. 239.

244. Rate of Change of Angular Momentum About a Line Through the Mass-center and Fixed in Direction. — Since this angular momentum does not depend on the motion of the mass-center, its rate of change also is independent of that motion. And one may calculate the rate as though the mass-center were fixed, by the methods explained in Art. 241.

EXAMPLE 1. Figure 415 represents a pair of car wheels which, suppose, are rounding a curve, whose center is toward the right, and moving toward you, the positive z-direction. The mass of the wheels and axle is M; the weight is W; the radius of gyration with respect to the axis of the axle is k; the radius of the wheels is r; the radius of curvature of the track is R; the speed of O is V. The wheels are driven by a horizontal force applied perpendicularly to the axle at O. It is required to determine the vertical reactions of the rails on the wheels and the pressure on the flange of the outer wheel.

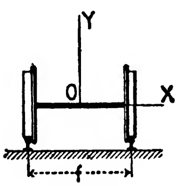


FIG. 415

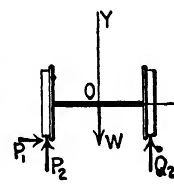


FIG. 416

Solution: For simplicity we suppose that the outer rail is not elevated and that the wheels are properly coned so that each one rolls perfectly without slipping, and we neglect rolling resistance. The forces acting on the wheels-and-axle are the applied force, the reactions P and Q of the rails, and W. In Fig. 416 we show these forces as far as possible. P and Q are regarded as replaced by their three axial components, but the x component of Q is obviously zero.

The x and y components of the acceleration of the mass-center O are

$$\bar{a}_x = V^2/R \quad \text{and} \quad \bar{a}_y = 0.$$

The components of the angular velocity ω of the wheels along OX, OY and OZ are

$$\omega_1 = V/r, \quad \omega_2 = V/R, \quad \text{and} \quad \omega_3 = 0.$$

The angular momentums about OX, OY and OZ are

$$h_1 = Mk^2V/r, \quad h_2 = I_2V/R, \quad h_3 = 0,$$

where I_2 denotes the moment of inertia of the wheel-and-axle about the y-axis. The only change in these angular momentums is the change in the direction of h_1 (O is now regarded as fixed). Hence

$$\frac{dh_x}{dt} = 0; \quad \frac{dh_y}{dt} = 0; \quad \frac{dh_z}{dt} = -Mk^2\frac{V}{r}\frac{V}{R}.$$

The first two of Eq. (1) and the last one of Eq. (2) Art. 235, furnish the desired solution. Thus

$$P_1 = M\frac{V^2}{R}, \quad P_2 + Q_2 - W = 0$$

and $$P_1r - P_2 \times \tfrac{1}{2}f + Q_2 \times \tfrac{1}{2}f = -Mk^2\frac{V^2}{Rr}.$$

The first equation gives P_1 directly, and simultaneous solution of the last two gives

$$P_2 = \frac{W}{2} + \frac{MV^2r}{Rf} + \frac{Mk^2V^2}{Rrf} \quad \text{and} \quad Q_2 = \frac{W}{2} - \frac{MV^2r}{Rf} - \frac{Mk^2V^2}{Rrf}.$$

The first terms in these two expressions are due to gravity. The second terms are due to centrifugal action or change in direction of motion of the mass-center; they have the same values as if the wheels were skidding, that is, they do not depend on the spin of the wheels. The third terms are due to gyrostatic action or change in direction of the axis of the rolling wheels. The components corresponding to the second terms and those corresponding to the third terms constitute couples, said to be due to centrifugal and gyroscopic action respectively. The moments of these couples are respectively

$$\frac{MV^2r}{R} \quad \text{and} \quad \frac{Mk^2V^2}{Rr}.$$

The ratio of the latter to the former is k^2/r^2; about 1/5 for a pair of locomotive driving wheels.

EXAMPLE 2. When an airplane is yawing, pitching or rolling there are "extra" reactions between the propeller and its shaft due to such motions. It is required to discuss the extra reaction due to a uniform yaw of an airplane flying at constant speed.

Solution: The reaction due to weight of the propeller and the air pressure on the propeller are to be omitted from the discussion. A rotating propeller of a yawing air plane on a flat plane in remote space is under the action of the forces which are to be discussed. Whatever these forces are, they may be compounded into a single force R at the center of the propeller and a couple C.

Let v denote the constant speed of the airplane; n the constant angular velocity of spin of the propeller, and N the constant angular velocity of yaw, both in radians per unit time. We suppose that the spin of the propeller (in front of the pilot) appears counter-clockwise, and that the yaw is toward his left. I_1, I_2 and I_3 are the principal moments of inertia of the propeller (Fig. 417). (No two of these are equal and in this respect this example differs from other similar ones.) Axes $O1$, $O2$ and $O3$ (the latter coinciding with the axis of the propeller shaft) are principal axes of the propeller. OA, OB and OC (the latter coinciding with $O3$) are fixed in the airplane. OX, OY and OZ (not shown) are fixed in direction but are taken so as to coincide with $OABC$ at the instant under consideration.

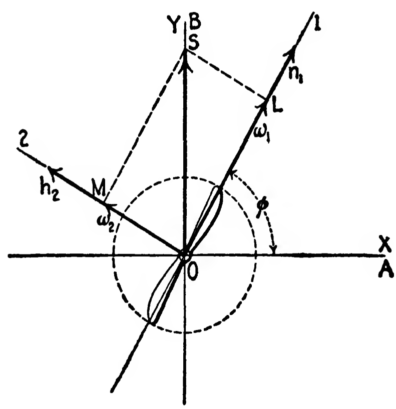


FIG. 417

The acceleration of the mass-center has these components,

$$\bar{a}_x = -\frac{v^2}{r}, \quad \bar{a}_y = 0, \quad \text{and} \quad \bar{a}_z = 0,$$

where r denotes the radius of curvature of the path of the mass-center where that point is at the instant in question. The axial components of R have these values,

$$R_x = -\frac{Mv^2}{r}, \quad R_y = 0, \quad \text{and} \quad R_z = 0,$$

where M denotes the mass of the propeller. R_x is directed toward the center of curvature; R does not depend on the yaw.

The angular velocity ω of the propeller is the resultant of n, about the propeller axis, and N, about the axis OB of yaw. Hence ω has these components

$$\omega_1 = N \sin \phi, \quad \omega_2 = N \cos \phi, \quad \text{and} \quad \omega_3 = n.$$

The coördinate frame $OABC$ has an angular velocity θ equal to N, about OB. Hence θ has these components

$$\theta_a = 0, \quad \theta_b = N, \quad \text{and} \quad \theta_c = 0.$$

The angular momentums of the propeller about the principal axes are

$$h_1 = I_1 N \sin \phi, \quad h_2 = I_2 N \cos \phi, \quad \text{and} \quad h_3 = I_3 n.$$

Resolving these along OA, OB and OC and combining properly, it will be found that

$$h_a = \tfrac{1}{2}(I_1 - I_2) N \sin 2\phi, \quad h_b = (I_1 \sin^2 \phi + I_2 \cos^2 \phi) N, \quad \text{and} \quad h_c = I_3 n.$$

Differentiation gives

$$\frac{dh_a}{dt} = (I_1 - I_2) Nn \cos 2\phi, \quad \frac{dh_b}{dt} = (I_1 - I_2) Nn \sin 2\phi, \quad \text{and} \quad \frac{dh_c}{dt} = 0.$$

Therefore (see Art. 241),

$$\frac{dh_x}{dt} = +(I_1 - I_2) Nn \cos 2\phi + I_3 Nn = C_x,$$

$$\frac{dh_y}{dt} = +(I_1 - I_2) Nn \sin 2\phi = C_y,$$

and

$$\frac{dh_z}{dt} = -\tfrac{1}{2}(I_1 - I_2) N^2 \sin 2\phi = C_z,$$

where C_x, C_y and C_z denote the components of C perpendicular to the x, y and z axes respectively. These components depend on the angular velocity of yaw, and on the angular position of the propeller. C_x is least for $\phi = 0$ and 180°, and greatest for $\phi =$ 90° and 270°. C_y and C_z are least for $\phi = 0$ and 90°, and greatest for $\phi = 45°$ and 135°. The greatest values of C_x, C_y and C_z respectively are

$$(I_3 + I_2 - I_1) Nn, \quad (I_2 - I_1) Nn, \quad \text{and} \quad \tfrac{1}{2}(I_2 - I_1) N^2.$$

C_x and C_y are exerted or furnished by the bearings of the crank (and propeller) shaft; C_z by the connecting rods. The shaft exerts couples equal and opposite to C_x, C_y and C_z on the bearings and the rods; these are the so-called "gyrostatic couple-reactions."

EXAMPLE 3. Figure 418 represents a fly wheel imperfectly mounted on a shaft OX, the axis of the wheel being inclined to the axis of the shaft. A wheel so mounted when rotating with the shaft is said to "wobble." Obviously it tends to set its own axis $O1$ into the axis of rotation OX, that is to decrease the angle between these axes. In so doing it tends to bend the shaft and the bearings react on the shaft to hold the ends of the shaft in place. Figure 419 shows the wheel in two extreme positions of wobble, the bent shaft (exaggerated) and the reactions of the bearings. It is required to develop a formula for the (centrifugal) torque exerted by the wheel on the shaft, which bends the shaft and induces the reactions of the bearings as just described.

Solution: Of course the shaft bends on account of the weight of the pulley but this bending is not under consideration now, that is it is neglected here. Or, one may imagine the wheel, shaft, and bearings (on a suitable base) removed to remote space and make the solution for the rotating wheel so placed. Let I_1 and I_2 be the moments of inertia of the wheel about its axis and about a diameter respectively, ω the angular velocity of rotation, and θ the angle between the axis of the wheel and the straight line joining the centers of the bearings. If the shaft is so stiff that it does not bend appreciably then θ is also the angle between the axis of the wheel and the (straight) axis of the shaft.

The components of ω along the principal axes $O1$, $O2$ and $O3$ are

$$\omega_1 = \omega \cos \theta, \quad \omega_2 = \omega \sin \theta, \quad \omega_3 = 0.$$

They are represented in the figure on the supposition that the wheel is rotating clockwise when viewed from the right-hand side. The angular momentums about the three principal axes are

$$h_1 = I_1 \omega \cos \theta, \quad h_2 = I_2 \omega \sin \theta, \quad \text{and} \quad h_3 = 0.$$

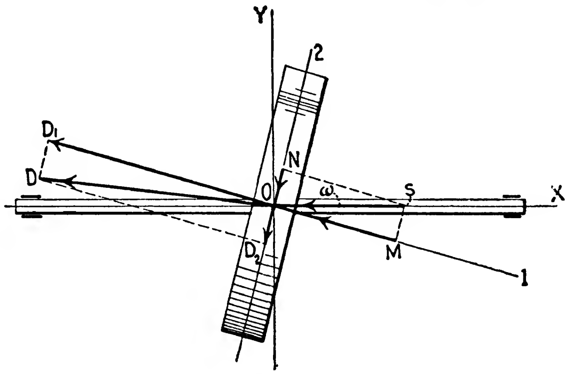


FIG. 418

OD_1 and OD_2 represent h_1 and h_2 respectively, and OD represents the angular momentum H of the wheel about O. The velocity of D is the rate of change of H (see Art. 238). D moves in a circle whose radius is

$$h_1 \sin \theta - h_2 \cos \theta = \tfrac{1}{2} (I_1 - I_2) \omega \sin 2\theta,$$

and the velocity of D is

$$\tfrac{1}{2} (I_1 - I_2) \omega^2 \sin 2\theta,$$

directed backward. Hence (see Art. 238),

$$\frac{dh_x}{dt} = 0, \quad \frac{dh_y}{dt} = 0, \quad \text{and} \quad \frac{dh_z}{dt} = -\tfrac{1}{2} (I_1 - I_2) \omega^2 \sin 2\theta.$$

Hence the reaction of the shaft on the (hub of the) wheel has no torques about the x- and y-axes, but a torque about the z-axis of

$$T_z = -\tfrac{1}{2} (I_1 - I_2) \omega^2 \sin 2\theta.$$

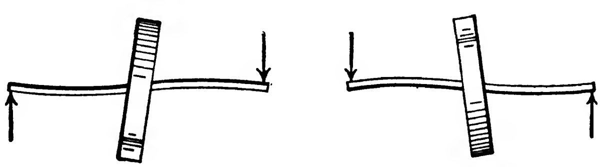


FIG. 419

The wheel exerts an equal and opposite (centrifugal) torque on the shaft. For a wheel whose rim is rectangular in cross section and so heavy that the spokes and hub are negligible, the foregoing formula becomes

$$M\left(\frac{D^2 + d^2}{32} - \frac{a^2}{24}\right)\omega^2 \sin 2\theta,$$

where M denotes mass of rim, D and d its outer and inner diameters, and a the width of its face. For moderate speeds and any ordinary value of θ, this moment is probably

of no consequence. Thus, its value is only 218 ft-lb. for the following cast-iron wheel: $D = 96$ in., $d = 76$ in., $a = 5\frac{1}{2}$ in., "wobble" $= \frac{1}{4}$ in. or $\theta = \tan^{-1}(0.25 \div 96)$, and speed $= 100$ rev/min.

245. Gyroscope; General Description. — The words gyroscope and gyrostat are generally used synonymously but sometimes a distinction is made, as follows: A gyrostat consists of a wheel and axle, both being symmetrical to the axis of the axle and mounted so that they may be rotated about that axis; a gyroscope consists of a gyrostat mounted in a frame which can be rotated. Figure 420 represents a common form of gyroscope; the gyrostat (wheel W and axle AA') is supported by a ring R which can be rotated about the axis BB'; the axle BB' is supported by the forked pillar F which can be rotated about the axle C, which is a part of the base. Thus the wheel can be rotated about its center into any desired position.

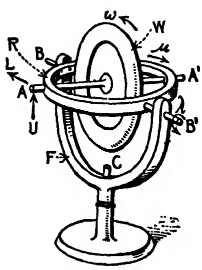


FIG. 420

The gyroscope seems to have been designed for illustrating principles of composition of rotations (Art. 148). In 1852 Foucault (French physicist) made an interesting application of the instrument; by its means he practically made visible the rotation of the earth. More recently the gyroscope has been made use of in several connections, — to steer a torpedo; to serve as a substitute, unaffected by the iron of the ship, for the ordinary (magnetic) mariner's compass; to stabilize a mono-rail car; and to steady a ship in a rough sea.

When its wheel is spinning, a gyroscope possesses properties which seem peculiar to the novice, inasmuch as it does not always respond as expected to efforts made to change its motion or position, as the following illustrations show. If a gyroscope like that represented in Fig. 420, well made and practically frictionless at all bearings and pivots, be grasped by the base and then moved about in any way, the axle of the wheel remains fixed in direction in spite of any attempt to alter it. The (gimbal) method of support makes it impossible to exert any resultant torque on the gyrostat (by way of the pillar) about any line through the center; and hence, there can be no change in the angular momentum of the gyroscope about any line through the center, which means that the direction of the spin axis AA' remains unchanged.

For another example, consider the effect of a torque applied directly to the gyrostat. A vertical force, say, applied at A would turn the gyrostat when not spinning about the axis B; but when the gyrostat is spinning, the force would rotate the spin-axis about the axis C, the direction of rotation depending upon the direction of spin. When the gyrostat is spinning in the direction indicated by the arrow ω, then an upward force U would rotate the spin-axis about C in the direction indicated by the

arrow μ. Again, a horizontal force applied at A, say, would turn the gyrostat when not spinning about the axis C; but when the gyrostat is spinning, the force would rotate the spin-axis about BB'. A force L would rotate the spin-axis in the direction indicated by the curved arrow λ.

This behavior of a spinning gyrostat under the action of torque is exhibited more strikingly by a gyroscope represented in Fig. 421. The wheel may be spun on the axle A; the gyrostat and its frame may be rotated about the axis BB'; and all may be rotated about the axis CC'. W is a weight which can be clamped on the stem A' to balance or unbalance the

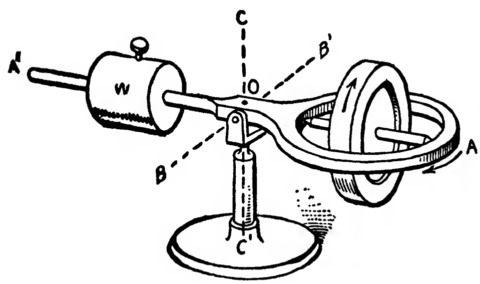


FIG. 421

frame with respect to the axis BB'. Now imagine W clamped so that the frame (with W and the gyrostat) is unbalanced. Then if the gyrostat be set spinning and the frame be released in the position shown, say, the frame will not rotate about BB' but about CC'. The direction of this rotation depends on the direction of spin and on the direction of the torque of gravity about BB'. If, for example, W is clamped quite near BB' so that the torque of gravity is clockwise as seen from B and the spin is as indicated, then A rotates toward B. This rotation persists except in so far as it is interfered with by friction at the pivots, and air resistance. (See *Spinning Tops* by Professor John Perry for more extensive simple descriptions of behavior of gyroscopes.)

Any such rotation of the axis of a spinning gyrostat is called a *precessional motion* or precession of the axis or of the gyrostat; the axis and the gyrostat are said to precess. We call precession *normal* or *oblique* according as the axis precesses about a line perpendicular or inclined to the axis.

It may not be clear from the foregoing examples of precession how to predict the direction of precession that would result from applying a given torque to a gyrostat with a given spin. The following is a simple rule; it is based on the dynamics of the whole matter as will be seen later: "When forces act upon a spinning body tending to cause rotation about any other axis than the spinning axis, the spinning axis sets itself in better agreement with the new (other) axis of rotation; perfect agreement would mean perfect parallelism, the direction of rotation being the same" (from

Spinning Tops). Or, what amounts to the same thing, *the precession is such as to turn the spin-vector*[1] *toward the torque-vector.*

There is another item of gyrostat behavior worth noting here. Suppose the gyrostat shown in Fig. 421 to be precessing as already explained. If the precession be hurried, say by means of a horizontal push applied at A', the center of gravity of the frame (with gyrostat and weight) rises; if the precession be retarded, the center of gravity descends. This behavior is in accordance with the rule for predicting precession. In the first case we have a torque about CC'; the torque vector is in the direction OC'; the spin-vector is in the direction OA'; and in accordance with the rule OA' turns toward OC', that is, the center of gravity rises. In the second case we have a torque about CC', the torque-vector being OC; and the spin-vector OA' turns toward OC, that is, the center of gravity descends. Thus we may state as another rule: *Hurry a precession, the gyrostat rises or opposes the torque which causes the precession; retard a precession, the gyrostat falls, or yields to the torque which causes the precession.*

246. Steady Normal Precession. — Let O, Fig. 422, be a ball-end of an axle on which a wheel is mounted; and suppose the ball-end rests in a fixed socket so that the wheel and axle can spin about OX, and OX can rotate about OY, a fixed axis. Let n be the angular velocity of spin and N that of the precession, both constant. Suppose that the spin is counterclockwise when viewed from the right, and the precession is counterclockwise when viewed from above (at Y). Then IO and JO may represent n and N respectively. The angular velocity ω of the gyrostat is the resultant of n and N; it is represented by SO. The components of ω along the principal axis $O1$, $O2$ and $O3$ (the latter not shown) are respectively

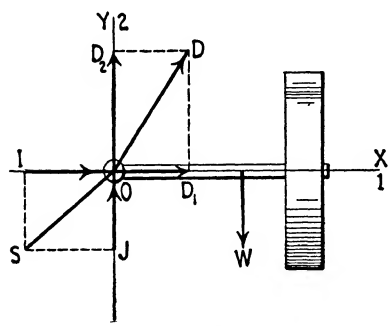


FIG. 422

$$\omega_1 = n, \quad \omega_2 = N \quad \text{and} \quad \omega_3 = 0.$$

[1] A *spin-vector* is a vector on the axis of spin, its arrow-head pointing to the place from which the spin appears counterclockwise; or — what amounts to the same thing — the arrow-head points in the direction along which the axis would advance if it were a right-hand screw turning in a fixed nut. The length of the vector — immaterial in the present connection — represents the angular velocity of spin to some convenient scale. Likewise the torque-vector is a vector parallel to the line about which the torque is taken and pointing to the place from which the rotation, which the torque tends to produce, would appear counterclockwise; or — what amounts to the same thing — the arrow-head points in the direction along which the vector would advance if it were a right-handed screw turned by the torque in a fixed nut.

The angular momentums of the gyrostat about the principal axes are

$$h_1 = I_1 n, \quad h_2 = I_2 N \quad \text{and} \quad h_3 = 0.$$

Hence OD_1 and OD_2 may represent h_1 and h_2; then OD represents H, the angular momentum of the gyrostat about O.

The rate at which H changes is the same as the velocity of D; hence

$$\frac{dH}{dt} = -h_1 N = -I_1 n N,$$

the negative sign being prefixed because the velocity of D is backward, the negative z direction. The components of this rate along the x, y and z axes respectively are

$$\frac{dh_x}{dt} = 0, \quad \frac{dh_y}{dt} = 0 \quad \text{and} \quad \frac{dh_z}{dt} = I_1 n N.$$

To maintain the assumed precession, the external forces acting on the gyrostat must be such that their torques about the x, y and z axes equal these rates respectively, that is,

$$T_x = 0, \quad T_y = 0, \quad \text{and} \quad T_z = -I_1 n N.$$

The sense of T_z must agree with that of the angular momentum which is being added. (Figure 423 shows the axes OX and OZ; OY is perpendicular to the paper. OD'_1 represents h_1 at a later instant than that considered above. Since h_2 does not change, $D_1D'_1$ represents the entire change in the angular momentum about O. As the interval is taken shorter and shorter, $D_1D'_1$ becomes more nearly parallel to ZO; hence the angular momentum which is being added when the spin axis coincides with OX is directed from Z to O.)

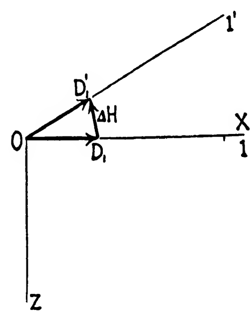


FIG. 423

Let W denote the weight of the gyrostat and l the distance from the center of gravity to the pivot, or center O. If the pivot is frictionless then the only torque of external forces about the axes x, y and z is $-Wl$, about OZ. Hence, the torque equations are satisfied if

$$Wl = I_1 n N.$$

247. Steady Oblique Precession. — Let ϕ denote the constant angle between the spin axis $O1$ (Fig. 424) and the axis OY about which $O1$ is precessing; n the angular velocity of spin, and N the angular velocity of precession, both constant. OX is a fixed axis perpendicular to OY, and chosen so that it is in the plane of $YO1$ at a certain instant. Axes OZ and $O3$ (not shown) are perpendicular to the paper.

The motion of the gyrostat consists of the component rotations n about $O1$ and N about OY. Let vectors $L'O$ and $M'O$ represent n and N re-

spectively; then the diagonal SO represents the resultant angular velocity ω of the gyrostat. The components of ω along the principal axes $O1$, $O2$ and $O3$ respectively are

$$\omega_1 = n + N\cos\phi, \quad \omega_2 = N\sin\phi \quad \text{and} \quad \omega_3 = 0.$$

Hence the angular momentums about these lines are

$$h_1 = I_1(n + N\cos\phi), \quad h_2 = I_2N\sin\phi, \quad \text{and} \quad h_3 = 0.$$

If OD_1 and OD_2 represent h_1 and h_2 respectively, then the diagonal OD represents H, the angular momentum of the gyrostat about O.

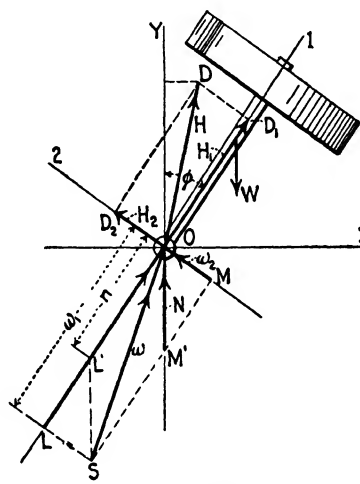


FIG. 424

The rate of change of the angular momentum of the gyrostat about O is the same as the velocity of D. The radius of the circle described by D is

$$h_1\sin\phi - h_2\cos\phi = I_1\omega_1\sin\phi - I_2N\sin\phi\cos\phi;$$

hence

$$dH/dt = -(I_1\omega_1\sin\phi - I_2N\sin\phi\cos\phi)N,$$

the negative sign being prefixed because the velocity of D is in the negative z direction. Therefore

$$\frac{dh_x}{dt} = 0, \quad \frac{dh_y}{dt} = 0, \quad \text{and} \quad \frac{dh_z}{dt} = \frac{dH}{dt}.$$

The torques of all forces acting on the gyrostat about the coördinate axes are equal to these rates respectively, that is

$$T_x = 0, \quad T_y = 0, \quad T_z = I_2N^2\sin\phi\cos\phi - I_1N\omega_1\sin\phi.$$

Therefore for steady spin and precession there must be no torque on the gyrostat about any line in the plane of the spin and precession axes, but a torque equal to

$$I_2N^2\sin\phi\cos\phi - I_1N\omega_1\sin\phi$$

about a line perpendicular to those axes and through the fixed point. And the sense of the torque and that of the increments in angular momentum must agree. For example, in Fig. 424, the sense of the torque must be clockwise.

We now consider whether a gyrostat may precess steadily under the influence of gravity W and the pivot reaction R only. Let l denote the distance from the center of gravity of the gyrostat to the pivot O. Then there is no torque of external forces about the axes OX or OY, but a torque $-Wl \sin \phi$ about OZ. If now the quantities W, l, ϕ, etc. have such values that

$$Wl \sin \phi = I_1 N \omega_1 \sin \phi - I_2 N^2 \sin \phi \cos \phi$$

then all the conditions for steady precession are satisfied. Evidently such values can be assigned in general.

Solving for N in the preceding equation gives

$$N = \frac{I_1\omega_1 \pm \sqrt{I_1^2\omega_1^2 - 4\,I_2 Wl \cos \phi}}{2\,I_2 \cos \phi},$$

from which it is plain that in general there are two possible velocities of precession N for a given gyrostat, spin velocity n, and obliquity ϕ. But if $I_1^2\omega_1^2 = 4\,I_2 Wl \cos \phi$, then there is only one value of N; if $I_1^2\omega_1^2 < 4\,I_2 Wl \cos \phi$, then N is imaginary, and the gyrostat will not precess.[1]

[1] Crabtree's *Spinning Tops and Gyroscopic Motion* is an excellent book for further study of that subject.

APPENDIX A

THEORY OF DIMENSIONS OF UNITS

A1. Dimensions of Units. — The magnitude of a quantity is expressed by stating how many times larger it is than a standard quantity of the same kind and naming the standard. Thus, we say that a certain distance is 10 miles, meaning that the distance is 10 times as great as the standard distance, the mile. The number expressing the relation between the magnitude of the quantity and the standard (the number 10 in the illustration) is called the *numeric* (or numerical value) of the quantity, and the standard is called the *unit.*

A unit for measuring any kind of quantity may be selected arbitrarily, but it must of course be a quantity of the same kind as the quantity to be measured. Thus, as unit of velocity we might select the velocity of light, as unit of area the area of one face of a silver dollar, etc. Many units in use are arbitrarily chosen, that is, without reference to another unit (for example, the bushel and the degree); but generally it is convenient practically to define them with reference to each other. All mechanical and nearly all physical quantities can be defined in terms of three arbitrarily selected units, not dependent on any other units. These are called *fundamental units,* and the others, defined with reference to them, *derived units.* It is customary in works on theoretical mechanics and physics to choose as fundamental the units of

length, mass, and *time;*

but it is sometimes more convenient to take as fundamental the units of

length, force, and *time.*

We give an analysis of derived units with reference to each of these sets of fundamentals, and two tables in which the absolute units are referred to the first set of fundamentals and the gravitational units to the second set. But either set might serve as fundamentals for all absolute and gravitational units.

A statement of the way in which a derived unit depends on the fundamental units involved in it is called a statement of the dimensions of the unit. For example,

$$\frac{\text{one square yard}}{\text{one square foot}} = \frac{(\text{one yard, or three feet})^2}{(\text{one foot})^2} = 9.$$

Thus, a unit of area depends only on the unit of length used, and the unit

of area varies as the square of the unit of length. This relation is expressed in the form of a "dimensional equation" as follows:

$$(\text{unit area}) = (\text{unit length})^2,$$

and briefly a unit area is said to be "two dimensions in length." Similarly, a unit volume is said to be three dimensions in length. We proceed to determine "dimensional formulas" for the units of several of the quantities of mechanics. The student should be able to determine formulas (see subsequent tables) for the others.

Velocity. — According to the definition of velocity (Art. 104), a unit velocity is directly proportional to the unit length and inversely to the unit time; hence if V, L and T denote units of velocity, length and time, respectively, the dimensional equation is

$$V = L/T = LT^{-1},$$

and a unit velocity is one dimension in length and minus one in time.

Acceleration. — According to the definition of acceleration (Art. 107), a unit acceleration is proportional directly to the unit velocity and inversely to the unit time; hence if A denotes unit acceleration, the dimensional equation is

$$A = V/T = L/T^2 = LT^{-2},$$

and a unit acceleration is one dimension in length and minus two in time.

Angular Velocity. — According to the definition of angular velocity (Art. 136), a unit angular velocity is proportional directly to the unit angle and inversely to the unit time; hence if ω and θ denote units of angular velocity and angle respectively, the dimensional equation is

$$\omega = \theta/T \quad \text{or} \quad \omega = T^{-1},$$

since units of angle (degree, radian, etc.) are independent of the fundamental units. A unit angular velocity is therefore minus one dimension in time.

Angular Acceleration. — According to the definition of angular acceleration (Art. 137), a unit angular acceleration is proportional directly to the unit angular velocity and inversely to the unit time; hence if α denotes uni; angular acceleration, the dimensional equation is

$$\alpha = \omega/T = T^{-2},$$

and a unit angular acceleration is minus two dimensions in time.

Force. — In accordance with the equation of motion of a particle (Art. 165), $F = ma$, or

"force = mass × acceleration;"

that is, the unit force is directly proportional to the units of mass and acceleration. Hence if F and M denote units of force and mass respectively, the dimensional equation is

$$F = MA = LMT^{-2},$$

ABSOLUTE SYSTEMS

Names of Quantities	Dimensional Formulas	Names of Units	
		C.G.S.	F.P.S.
Length	L	centimeter (cm)	foot (ft)
Mass	M	gram (gr)	pound (lb)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	cm/sec ("kine")	ft/sec
Acceleration	LT^{-2}	cm/sec^2 ("spoud")	ft/sec^2
Angular velocity	T^{-1}	rad/sec	rad/sec
Angular acceleration	T^{-2}	rad/sec^2	rad/sec^2
Force	LMT^{-2}	dyne	poundal (pdl)
Weight	LMT^{-2}	dyne	pdl
Moment of mass	LM	gr-cm	lb-ft
Moment of inertia (body)	L^2M	gr-cm^2	lb-ft^2
Moment of force	L^2MT^{-2}	cm-dyne	ft-pdl
Work	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Energy	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Power	L^2MT^{-3}	erg/sec	ft-pdl/sec
Impulse	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Momentum	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Density	$L^{-3}M$	gr/cm^3	lb/ft^3
Specific weight	$L^{-2}MT^{-2}$	dyne/cm^3	pdl/ft^3
Moment of area	L^3	cm^3	ft^3
Moment of inertia (area)	L^4	cm^4	ft^4
Stress	LMT^{-2}	dyne	pdl
Stress intensity	$L^{-1}MT^{-2}$	dyne/cm^2	pdl/ft^2

and a unit force is one dimension in length, one in mass, and minus two in time.

Mass. — If we regard length, force, and time as fundamental units, then the last equation written as follows is the dimensional equation for a unit mass:

$$M = FT^2/L = L^{-1}FT^2,$$

and a unit mass is minus one dimension in length, one in force, and two in time.

Work. — According to the definition of work (Art. 192), the unit of work is directly proportional to the units of force and length; hence if W denotes unit work, the dimensional equation is

$$W = LF = L^2MT^{-2},$$

and a unit work is one dimension in length, one in force, or two in length, one in mass, and minus two in time.

Power. — According to the definition of power (Art. 201), a unit of power is proportional directly to the unit work and inversely to the unit time; hence if P denotes unit of power, the dimensional equation is

$$P = W/T = LFT^{-1} = L^2MT^{-3},$$

and a unit power is one dimension in length and force and minus one in time, or two in length, one in mass, and minus three in time.

GRAVITATION SYSTEMS

Names of Quantities	Dimensional Formulas	Names of Units	
		F.P. (force) S.	M.K. (force) S.
Length	L	foot (ft)	meter (m)
Force	F	pound (lb)	kilogram (kg)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	ft/sec	m/sec
Acceleration	LT^{-2}	ft/sec^2	m/sec^2
Angular velocity	T^{-1}	rad/sec	rad/sec
Angular acceleration	T^{-2}	rad/sec^2	rad/sec^2
Mass	$L^{-1}FT^2$	"slug" (sl)	"metric slug" (msl)
Weight	F	lb	kg
Moment of mass	FT^2	sl-ft	msl-m
Moment of inertia..(body)	LFT^2	sl-ft^2	msl-m^2
Moment of force	LF	ft-lb	kg-m
Work	LF	ft-lb	kg-m
Energy	LF	ft-lb	kg-m
Power	LFT^{-1}	ft-lb/sec	kg-m/sec
Impulse	FT	lb-sec	kg-sec
Momentum	FT	lb-sec	kg-sec
Density	$L^{-4}FT^2$	sl/ft^3	msl/m^3
Specific weight	$L^{-3}F$	lb/ft^3	kg/m^3
Moment of area	L^3	ft^3	m^3
Moment of inertia..(area)	L^4	ft^4	m^4
Stress	F	lb	kg
Stress intensity	$L^{-2}F$	lb/ft^2	kg/m^2

A2. Applications of the Theory of Dimensions. — A knowledge of the theory of dimensions is probably of most value to the beginner as a help to a clear understanding of the different mechanical quantities and the relations between them. The theory is useful practically in other ways, three of which we mention.

(1) *As a test of the accuracy of equations between mechanical quantities.* — Such an equation if *rationally* and *correctly* deduced must be homogeneous, that is the terms in it must be the same in kind. To ascertain whether terms are the same in kind we write the dimensional form of the equation, reduce the terms to their simplest forms and compare; if they are alike, the terms are the same in kind. To illustrate let us consider equation (4), Art. 95,

$$T = \frac{1}{2}wa\left(1 + \frac{a^2}{16f^2}\right)^{\frac{1}{2}},$$

where T denotes tension (or force), w weight (or force) per unit length, and a and f lengths. Using L, M and T, the dimensional form of the equation is

$$LMT^{-2} = \frac{LMT^{-2}}{L}L\left(\frac{L^2}{L^2}\right)^{\frac{1}{2}}.$$

Since the right-hand member reduces to LMT^{-2}, the two members are alike in kind, as they should be. The coefficient $\frac{1}{2}$ and the term 1 were

omitted from the dimensional equation because they are independent of L, M and T. Using L, F and T, the dimensional form of the equation is

$$\mathrm{F} = \frac{\mathrm{F}}{\mathrm{L}}\mathrm{L}\left(\frac{\mathrm{L}^2}{\mathrm{L}^2}\right)^{\frac{1}{2}},$$

which is simpler than the first form. Indeed dimensional equations based on L, F and T are generally the simpler in the case of formulas with which engineers have to deal, particularly if mass does not appear in the formula.

Showing that an equation is homogeneous does not prove that it is correct, but that it may be correct; showing that an equation is non-homogeneous shows it to be incorrect. Since abstract numbers do not appear in the dimensional form of an equation, the test for homogeneity does not disclose errors in numerical coefficients and terms, nor errors in signs.

(2) *To express a magnitude in different units.* — Obviously the numerical value of a given quantity changes inversely as the magnitude of the unit used; thus a certain distance may be expressed as

10 mi., 17,600 yds., and 52,800 ft.,

and plainly the numerics are respectively as 1, 1760 and 5280, while the corresponding units are as 5280, 1760 and 1.

Let q_2 be the known numerical value of a quantity when expressed in the unit Q_2, and q_1 the numeric (to be found) of the same quantity expressed in the unit Q_1; then

$$q_1/q_2 = Q_2/Q_1, \quad \text{or} \quad q_1 = q_2Q_2/Q_1.$$

The ratio $\dfrac{Q_2}{Q_1}$ can be easily computed by substituting for Q_1 and Q_2 their *equivalents* in terms of fundamental units; thus if a, b and c are the dimensions of Q_1 (and Q_2),

$$Q_1 = k_1(\mathrm{L}_1^a\mathrm{M}_1^b\mathrm{T}_1^c) \quad \text{and} \quad Q_2 = k_2(\mathrm{L}_2^a\mathrm{M}_2^b\mathrm{T}_2^c),$$

where L_1, M_1 and T_1 are the *particular* fundamentals for Q_1; L_2, M_2 and T_2 those for Q_2; and k_1 and k_2 numerical coefficients (very often unity). Finally,

$$q_1 = q_2\frac{k_2}{k_1}\left(\frac{\mathrm{L}_2}{\mathrm{L}_1}\right)^a\left(\frac{\mathrm{M}_2}{\mathrm{M}_1}\right)^b\left(\frac{\mathrm{T}_2}{\mathrm{T}_1}\right)^c.$$

As an example, let us determine how many watts in 10 horse-power. Since Q_1 (horse-power) = 550 ft-lb-sec^{-1}, and Q_2 (watt) = 10^7 ergs per sec. = 10^7 cm-dyne-sec^{-1},

$$q_1 = 10\frac{550}{10^7}\frac{\text{ft}}{\text{cm}}\frac{\text{lb}}{\text{dyne}}\frac{\text{sec}^{-1}}{\text{sec}^{-1}} = 10\frac{550}{10^7}(30.48)\ (4.45 \times 10^5)\ (1) = 7460.$$

(3) *To ascertain the unit of the result of a numerical calculation.* — Substitute for the quantities the names of the units in which they are expressed, and then repeat the calculation, treating the names as though they were

algebraic quantities. The reduced answer is the name of the unit of the numerical answer. Thus in the formula for the elongation of a rod due to a pull at each end, Pl/AE (wherein P denotes pull, l length of the rod, A area of cross-section, and E Young's modulus for the material), suppose that $P = 10{,}000$ lb., $l = 50$ in., $A = 0.5$ in², $E = 30{,}000{,}000$ lb/in²; the calculations for elongation and name of unit are

$$\frac{10{,}000 \times 50}{0.5 \times 30{,}000{,}000} = 0.33, \quad \text{and} \quad \frac{\text{lb.} \times \text{in.}}{\text{in}^2 \times \text{lb/in}^2} = \frac{\text{lb.} \times \text{in.} \times \text{in}^2}{\text{in}^2 \times \text{lb.}} = \text{in.}$$

APPENDIX B[1]

Moment of Inertia and Radius of Gyration

B1. General Principles, Etc. — Perhaps every student has observed that the effort required to start a body to rotating about a fixed axis seems to depend not only on the mass of the body but also on the remoteness of the material of the body from the axis of rotation. Figure B1 represents a simple apparatus by means of which one can roughly "sense" this fact. It consists of a vertical shaft S to which a grooved pulley P and cross arm A are fastened rigidly, and a heavy body B which can be clamped on the cross arm. The pull or turning effort may be applied by means of a cord wrapped about the pulley. It is shown in Art. 177 that this "rotational inertia" of a body is proportional to the "moment of inertia" of the body about the axis, which may be defined as follows:

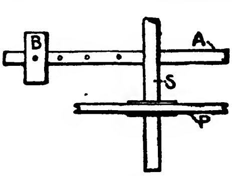


FIG. B1

The moment of inertia of a body with respect to a line is the sum of the products obtained by multiplying the mass of each particle of the body by the square of its distance from the line. Or, if I = moment of inertia, m_1, m_2, m_3, etc. = the masses of the particles, and r_1, r_2, r_3, etc., their distances respectively from the line or axis, then

$$I = m_1r_1^2 + m_2r_2^2 + \cdots = \Sigma mr^2;$$

or if the body is continuous, then

$$I = \int dM \cdot r^2, \quad \ldots \ldots \ldots \quad (1)$$

where dM denotes the mass of any elementary portion and r its distance from the line about which the moment of inertia is taken.[2]

[1] Moment of Inertia of Bodies and of Areas is treated in works on Integral Calculus nowadays, principally because the subject affords good practical exercise in integration. Appendix B is included in this book for the student who may not have had adequate training in the subject for the understanding of certain other parts of this book in Chaps. XI and XIV. Students are cautioned to distinguish carefully between moments of inertia of (physical) *bodies* and of (geometrical) *areas*. The former are employed in Dynamics; the latter in "Strength of Materials," or "Elasticity," and to a lesser extent in other branches of Applied Mechanics.

[2] Euler (1707–83) introduced the term "moment of inertia," and he explained its appropriateness (in his "Theoria Motus Corporum Solidorum," p. 167) somewhat as follows: The choice of the name, moment of inertia, is based on analogies in the equations of motion for translations and rotations. In a translation the acceleration is

In applying Equation (1), it is necessary to choose the elementary mass dM of such shape that all points of it will be equally distant from the axis or plane about which moment of inertia is being calculated so that there will be no doubt as to the meaning of r. (See Ex. 1 following.) In the case of a right prism of any cross section, the moment of inertia about a line parallel to the axis of the prism may be calculated perhaps most simply by choosing for the elementary mass a filament parallel to the axis and extending from base to base; then $dM = (adA)\delta$ where a is the altitude of the prism, dA = the cross section of the filament, and δ the density; and

$$I = a\delta \int dAr^2.$$

(See Ex. 2 following.) This integral, extending over the area of the cross section, is called the moment of inertia of the cross section about the line specified.

In some cases one may conceive the body as consisting of an infinite number of parts whose moments of inertia about the line or axis specified are known from previous calculation. Then

$$I = dI_1 + dI_2 + \cdot \cdot \cdot \cdot \cdot \cdot = \int dI.$$

(See Ex. 3 following.)

A unit of moment of inertia depends upon the units of mass and distance used. There is no single-word name for any unit of moment of inertia. Each unit is described by stating the units of mass and distance involved in it, and in accordance with the "make-up" of the unit.[1] Thus, when the pound and the foot are used as units of mass and length respectively, then the unit of moment of inertia is called a pound-foot square (lb-ft.^{2}); when the slug and the foot are used, then the unit moment of inertia is called slug-foot square (sl-ft.^{2}).

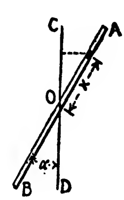


FIG. B2

EXAMPLE 1. Required to show that the moment of inertia of a slender rod about a line through the center is $\frac{1}{12} Ml^2 \sin^2 \alpha$, where M is the mass of the rod, l its length, and α the angle between the axis of the rod and the line.

Solution: Let a be the cross section of the rod, δ its density, and x = the distance of any elementary portion from the middle of the rod AB (Fig. B2). Then dM =

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proportional directly to the "accelerating force" and inversely to the mass, or "inertia," of the moving body; and in a rotation the angular acceleration is proportional directly to the moment of the accelerating force and inversely to a quantity, Σmr^2, depending on the mass or inertia. This quantity, to complete a similarity, we may call "moment of inertia." Then we have for translations and rotations respectively,

linear acceleration = (force)/(inertia or mass); and
angular acceleration = (moment of force)/(moment of inertia).

[1] For dimensions of a unit of moment of inertia, see Appendix A.

$\delta(a\,dx)$, and the distance of the element from $CD = x \sin \alpha$. Hence

$$I = \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \delta a\, dx \cdot x^2 \sin^2 \alpha = \delta a \sin^2 \alpha \left[\frac{x^3}{3}\right]_{-\frac{1}{2}l}^{+\frac{1}{2}l} = \frac{\delta a \sin^2 \alpha}{3}\,\frac{l^3}{4},$$

and this reduces to $\frac{1}{12} Ml^2 \sin^2 \alpha$, since $\delta al = M$.

EXAMPLE 2. Show that the moment of inertia of a very thin circular homogeneous disc about a diameter is $\frac{1}{4} MR^2$ where M is the mass of the disc and R its radius.

Solution: Let t be the thickness and δ the density of the disc; then (see Fig. B3) $dM = \delta t \cdot \rho \cdot d\theta\, d\rho$ and with reference to the diameter XOX, r of Eq. (1) is $\rho \sin \theta$. Hence,

$$I_x = \int_0^R \int_0^{2\pi} (\delta t \rho\, d\theta\, d\rho)\,(\rho \sin \theta)^2 = \delta t \int_0^R \int_0^{2\pi} \rho^3 \sin^2 \theta\, d\rho\, d\theta = \text{etc.}$$

EXAMPLE 3. Required to show that the moment of inertia of a right parallelopiped about a central axis parallel to an edge is $\frac{1}{12} M (a^2 + b^2)$ where M is the mass of the parallelopiped and a and b are the lengths of the edges which are perpendicular to that axis. See Fig. B4 where the z-axis is the one to which this moment of inertia corresponds.

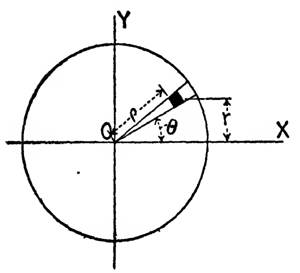

FIG. B3

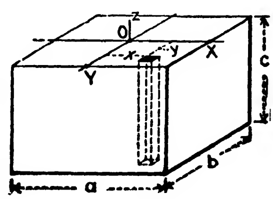

FIG. B4

Solution: We take for elementary portion a volume $dx\, dy\, dz$; its mass $= \delta(dx\, dy\, dz)$, and the square of its distance from the z-axis $= x^2 + y^2$. Hence

$$I = \delta \int_{-a/2}^{+a/2} \int_{-b/2}^{+b/2} \int_0^c (x^2 + y^2)\, dx\, dy\, dz = \frac{\delta c}{12}(a^3 b + ab^3) = \text{etc.}$$

EXAMPLE 4. Required to show that the moment of inertia of a right circular cylinder with respect to its axis is $\frac{1}{2} Mr^2$, where M is the mass of the cylinder and r the radius of its base.

Solution: We use the special method for prisms mentioned and choose polar coordinates (see Fig. B5); then

$$dA = \rho\, d\theta\, d\rho \text{ and } dM = \delta\,(a\rho\, d\theta\, d\rho); \text{ hence}$$

$$I = a\delta \int dA\, \rho^2 = a\delta \int_0^r \int_0^{2\pi} \rho^3\, d\rho\, d\theta = \frac{a\delta r^4\, 2\pi}{4} = \text{etc.}$$

EXAMPLE 5. Required to show that the moment of inertia of a sphere about a diameter is $\frac{2}{5} Mr^2$ where $M =$ its mass and $r =$ its radius.

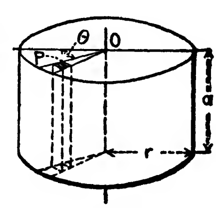

FIG. B5

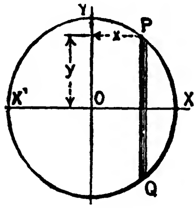

FIG. B6

Solution: We might begin with Eq. (1) but we use the special method for prisms, making use of the result found in Ex. 4. Thus we conceive the sphere made of laminas perpendicular to the diameter in question; determine the moment of inertia of each lamina; and then add the moments of inertia of the laminas. Let XX' (Fig. B6) be the diameter in question, and PQ a section of one of the laminas; then the mass of the lamina is $\delta(\pi y^2 dx)$. According to Ex. 4 the moment of inertia of this lamina (cylinder) about its axis (XX') is $\frac{1}{2}\,\delta(\pi y^2 dx) y^2$. Hence the moment of inertia of the sphere is

$$\int_{-r}^{+r} \tfrac{1}{2}\,\delta(\pi y^4 dx) = \tfrac{1}{2}\,\pi\delta \int_{-r}^{+r} (r^2 - x^2)^2\, dx = \tfrac{8}{15}\,\delta\pi r^5 = \text{etc.}$$

B2. Radius of Gyration. — Since a moment of inertia is one dimension in mass and two in length, it can be expressed as the product of a mass and a length squared; it is sometimes convenient to so express it. The *radius of gyration* of a body with respect to a line is such a length whose square multiplied by the mass of the body equals the moment of inertia of the body with respect to that line. That is, if k and I denote the radius of gyration and moment of inertia of the body with respect to any axis and M = its mass, then

$$k^2M = I \quad \text{or} \quad k = \sqrt{I/M}.$$

The radius of gyration may be regarded as follows: If one imagines all the material of a body concentrated into a point so located that the moment of inertia of the material point about the line in question equals the moment of inertia of the body about that line, then the distance between the line and the point equals the radius of gyration of the body about that line. The material point is sometimes called the *center of gyration* of the body for the particular line.

To furnish still another view of radius of gyration we call attention to the fact that the square of the radius of gyration of a homogeneous body with respect to any line is the mean of the squares of the distances of all the equal elementary parts of the body from that line. For let r_1, r_2, etc., be the distances from the elements, dM, to the axis, and let n denote their number (infinite). Then the mean of the squares is

$$\frac{r_1^2 + r_2^2 + \cdots\cdots}{n} = \frac{r_1^2dM + r_2^2dM + \cdots\cdots}{ndM} = \frac{I}{M};$$

and since $\quad \dfrac{I}{M} = k^2, \quad k^2 = \dfrac{r_1^2 + r_2^2 + \cdots\cdots}{n}.$

Obviously the radius of gyration of a body with respect to a line is intermediate between the distances from the line to the nearest and most remote particles of the body. This fact will assist in *estimating* the radius of gyration of a body.

B3. Parallel Axis Theorem; Reduction or Transformation Formulas. — There is a simple relation between the moments of inertia (and the radii of gyration) of a body with respect to parallel lines one of which passes through the mass-center of the body. By means of this relation one can simplify many calculations of moment of inertia, and avoid integrations (see examples following); it may be stated as follows:

The moment of inertia of a body with respect to any line equals its moment of inertia with respect to a parallel line passing through the mass-center plus the product of the mass of the body and the square of the distance between the lines. Or, if I = the first moment of inertia, $\bar{I}$ = the second (for the line through the mass-center), M = mass, and d = the distance between the parallel lines,

$$I = \bar{I} + Md^2. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Proof. — Let O (Fig. B7) be the mass-center, and P any other point of the body (not shown), LL the line about which the moment of inertia is I, and OZ a parallel line (through the mass-center) about which the moment of inertia is $\bar{I}$. The distance between these parallel lines is d. For convenience we take x and y axes through O, the former in the plane of the two parallel lines and the latter perpendicular to that plane. Let x, y and z be the coördinates of P. The square of the distance of P from the z-axis

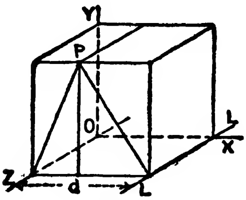

FIG. B7

equals $x^2 + y^2$, hence $\bar{I} = \int dM(x^2 + y^2)$. The square of the distance of P from the line LL equals $(d - x)^2 + y^2$, hence

$$I = \int [(d - x)^2 + y^2]\, dM = \int (x^2 + y^2)\, dM + d^2 \int dM - 2\,d \int x\, dM.$$

Now the first of the last three integrals $= \bar{I}$, and the second one $= Md^2$. If now we show that the third $= 0$, then formula (1) is proved. The third integral is proportional to the moment of the body with respect to the yz plane; but this plane contains the mass-center, and hence the moment equals zero (Art. 91). Thus, if W = weight of the body,

$$\int x\, dM = \int x \frac{dW}{g} = \frac{W\bar{x}}{g},$$

which is equal to zero since $\bar{x} = 0$.

The square of the radius of gyration of a body with respect to any line equals the square of its radius of gyration with respect to a parallel line passing through the mass-center plus the square of the distance between the two lines, or, if k = the first radius of gyration, $\bar{k}$ = the second, and d = the distance between the lines, then

$$k^2 = \bar{k}^2 + d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation results at once by dividing Eq. (1) by M.

According to (2) k is always greater than d; that is, *the radius of gyration of a body with respect to a line is always greater than the distance from the line to the center of gravity of the body.* But, if the dimensions of the cross sections of the body perpendicular to the line in question are small compared to d, then $\bar{k}/d$ is small compared to 1, and k equals d approximately (see Ex. 2). In such a case the moment of inertia is approximately equal to Md^2.

EXAMPLE 1. Required the moment of inertia of a prism of cast iron (weighing 450 lb/ft³.) 6 by 9 in. by 3 ft. with respect to one of the long edges.

Solution: The block weighs 507 lb.; hence according to Ex. 3, Art. B1, the moment of inertia of the block with respect to the line through the mass-center parallel to the long edge is $507\,(6^2 + 9^2) \div 12 = 4940$ lb-in.². The square of the distance from a

long edge to the mass-center is 29.25 in.^{2}; hence according to Eq. (1) the moment of inertia desired is $4940 + 507 \times 29.25 = 19{,}760$ lb-in.^{2} = 137 lb-ft.^{2} = 4.27 sl-ft.^{2}.

EXAMPLE 2. Required the radius of gyration of a round steel rod 1 in. in diameter with respect to a line 12 in. from the axis of the rod.

Solution: According to Ex. 4, Art. B1, the square of the radius of gyration of the rod with respect to its axis is $\frac{1}{2}\,0.5^2 = 0.125$ in.^{2}. According to Eq. (2) the radius of gyration desired is

$$\sqrt{(0.125 + 144)} = 12.01,$$

nearly the same as the distance from the line of reference to the mass-center of the rod.

EXAMPLE 3. It is required to show that the moment of inertia of a right circular cone with respect to a line through the apex and parallel to the base $= \frac{3}{20}\,M(r^2 + 4\,a^2)$ where M is the mass of the cone, r the radius of its base, and a its altitude.

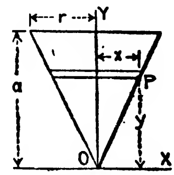

FIG. B8

Solution: We conceive the cone as made of laminas parallel to the base, find the moment of inertia of each lamina with respect to the specified line, and then add all the moments. For convenience we take the axis of the cone as the y coördinate axis, and the line for which the moment of inertia is required as the x-axis (Fig. B8). The moment of inertia of the lamina indicated about a diameter is $\frac{1}{4}\,dM \cdot x^2$ where dM = the mass of the lamina and x = its radius. Hence its moment of inertia about the x-axis $= \frac{1}{4}\,dMx^2 + dMy^2$ (see Eq. 1), and the moment of inertia of the entire cone $= \int (\frac{1}{4}\,dMx^2 + dMy^2)$, the limits being assigned so as to include all laminas. We choose to integrate with respect to y, and so must express dM and x in terms of y. From similar triangles in the figure $x/y = r/a$, or $x = ry/a$; obviously $dM = \delta\pi x^2 dy = \delta\pi(r^2y^2/a^2)\,dy$ where δ = density. Hence

$$I = \int_0^a \frac{\pi r^4 \delta y^4 dy}{4\,a^4} + \int_0^a \frac{\pi r^2 \delta y^4 dy}{a^2} = \frac{\pi r^4 \delta a}{20} + \frac{\pi r^2 \delta a^3}{5} = \text{etc.}$$

B4. Composite Body. — By this term is meant a body which one naturally conceives as consisting of finite parts, for example, a flywheel which consists of a hub, several spokes, and a rim. The moment of inertia of such a body with respect to any line can be computed by adding the moments of inertia of all the component parts with respect to that same line. The radius of gyration of a composite body does not equal the sum of the radii of gyration of the component parts. It can be determined from the equation at the top of page 305, where I = moment of inertia of the whole body and M = its mass.

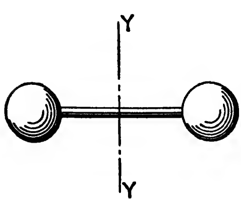

FIG. B9

EXAMPLE. It is required to determine the moment of inertia of the dumb-bell shown in Fig. B9 about the central axis YY. The cast-iron spheres are 6 in. in diameter and weigh 30 lbs. each; the steel handle — a relatively slender rod — is 16 in. long and weighs 1 lb.

Solution: The moment of inertia of either sphere about an axis through its center is $\frac{2}{5} \times 30 \times 3^2 = 108$ lb-in.^{2} (Ex. 5, Art. B1). The distance from the mass-center of either sphere to the axis YY is 11 in., therefore the moment of inertia of either sphere about YY is $108 + (30 \times 11^2) = 3738$ lb-in.^{2}. The moment of inertia of the handle about YY is $\frac{1}{12} \times 1 \times 16^2 = 21.3$ lb-in.^{2} (Ex. 1, Art. B1). The moment of inertia

of the entire dumb-bell about the axis YY is therefore $(2 \times 3738) + 21.3 = 7497$ lb-in.^{2} = 52.0 lb-ft.^{2} = 1.61 sl-ft.^{2}.

B5. Product of Inertia. — In equations pertaining to rotation (Art. 188) and in an important equation of the following article there are certain terms which we now name, explain and discuss briefly.

By product of inertia of a body with respect to a pair of coördinate planes is meant the sum of all products obtained by multiplying each elementary mass of a body by its coördinates with respect to the two planes, thus

$$K_{xy} = \Sigma dm \cdot xy, \quad K_{yz} = \Sigma dm \cdot yz, \quad K_{zx} = \Sigma dm \cdot zx$$

where K_{xy} is our symbol for the product of inertia with respect to the planes from which the x and y coördinates of dM are measured (the planes YOZ and ZOX); similarly K_{yz} and K_{zx}.

A product of inertia is the same *kind* of a quantity as a moment of inertia, that is, both are expressible in the same unit. Unlike a moment of inertia, a product of inertia may be negative or even zero.

It can be proved that for any body and for any origin of coördinates there is one set of three coördinate planes such that for each pair of those planes the product of inertia is zero. We do not prove this proposition nor show how to locate these planes except in a special common case: if a body is homogeneous and has a plane of symmetry then the product of inertia of the body is zero with respect to that plane and any other plane perpendicular to it.

Let the plane of symmetry be the plane YOZ, then for any particle whose coördinates are x, y, z there is another whose x coördinates are $-x$, y, z. Hence, the product of inertia of this *pair* of particles with respect to the plane of symmetry and, say, the ZOX plane is

$$mxy + m(-x)y = 0.$$

And hence the product of inertia of all pairs of particles, that is, of the entire body, is zero.

B6. Principal Axes. — We now examine the moments of inertia of a body about all lines through any point of it and develop an inclined axis formula. And we show that in general there is one line about which the moment of inertia is maximum, and a second line, perpendicular to the first, about which the moment of inertia is a minimum. These two lines and the one perpendicular to their plane at the point in question are called the *principal axes* at the point, and the moments of inertia about those lines are the *principal moments of inertia* at the point.

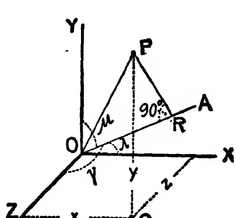

FIG. B10

Inclined Axis Formula. — Let P (Fig. B10) be a particle of a body (not otherwise shown); O any other point, not in the body necessarily; and OA any line through O.

Let λ, μ and ν = the direction cosines of OA with respect to any coördinate axes with origin at O, m = mass of P, r = distance of P from OA, and I = the moment of inertia of the body about OA. According to definition, $I = \Sigma mr^2$. Now $r^2 = (OP)^2 - (OR)^2$. OP is a diagonal of a parallelopiped of which lines x, y and z are three intersecting edges; hence $OP^2 = x^2 + y^2 + z^2$. OR is one side of the closed (gauche) polygon $OZQPRO$; and since any side of a closed polygon equals the algebraic sum of the projections of the other sides upon it, $OR = \lambda x + \mu y + \nu z + 0$. Hence

$$r^2 = x^2 + y^2 + z^2 - (\lambda x + \mu y + \nu z)^2.$$

Expanding this expression for r^2 and arranging terms we would find that $I = \Sigma m[\lambda^2(y^2 + z^2) + \mu^2(z^2 + x^2) + \nu^2(x^2 + y^2) - 2\,\mu\nu yz - 2\,\nu\lambda zx - 2\,\lambda\mu xy]$. In this (space) summation λ, μ and ν are constants; hence

$$I = \lambda^2\Sigma m(y^2 + z^2) + \cdot\cdot\cdot\cdot - 2\,\mu\nu\Sigma myz - \cdot\cdot\cdot\cdot$$

Now $y^2 + z^2$, $z^2 + x^2$, and $x^2 + y^2$ respectively = the squares of the distances of P from the x, y and z axes; hence if A, B and C = the moments of inertia of the body with respect to the x, y and z axes, we have

$$A = \Sigma m(y^2 + z^2), \quad B = \Sigma m(z^2 + x^2), \quad \text{and} \quad C = \Sigma m(x^2 + y^2).$$

The remaining summations in the foregoing expression for I are the products of inertia of the body with respect to the two coördinate planes intersecting in the x, y and z axes respectively. Let D, E and F respectively denote these products of inertia, that is

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad \text{and} \quad F = \Sigma mxy.$$

Then we have

$$I = \lambda^2 A + \mu^2 B + \nu^2 C - 2\,\mu\nu D - 2\,\nu\lambda E - 2\,\lambda\mu F. \quad . \quad . \quad . \qquad (1)$$

If we know the moments of inertia (A, B and C) of a body about each one of a set of coördinate axes, and the products of inertia (D, E and F) with respect to each pair of the coördinate planes, then by means of formula (1) we can find the moment of inertia I of the body about any line through the origin of coördinates. And by means of formula (1) of Art. B3 we can transfer this I to any parallel axis desired. Thus the two formulas enable one to "transfer" from the coördinate axes to any line whatsoever.

Principal Axes. — We prove that there are such axes as defined in the opening paragraph, by means of the *momental ellipsoid* now explained.

Imagine a length OS laid off on OA (Fig. B10) so that OS, which we will call ρ, is inversely proportional to the radius of gyration of the body about OA. That is, if k = the radius of gyration and Q a factor of proportionality, then $\rho = Q/k$. Such points S for all lines OA would lie on the surface of an ellipsoid (proved presently) called the *momental ellipsoid* of the body for the selected point O. Let X, Y and Z = the coördinates

of S; they equal $\rho\lambda$, $\rho\mu$ and $\rho\nu$ respectively. Then equation (1) multiplied by ρ^2 reduces to

$$AX^2 + BY^2 + CZ^2 - 2\,DZY - 2\,EZX - 2\,FXY = Q^2M, \quad . \quad (2)$$

where M = the mass of the body. This is the equation of an ellipsoid with center at O (see any standard work on Analytic Geometry). The momental ellipsoid might of course be one of revolution in a special case, or even a sphere.

In general, the axes of an ellipsoid are unequal in length. Hence the radius of gyration (and the moment of inertia) about the shortest axis of the momental ellipsoid is greater than the radius of gyration (and moment of inertia) about any other line through the center of the ellipsoid, and the moment of inertia about the longest axis is less than that about any other line through the center. Thus we have shown that there *are* two lines at right angles to each other through any point of a body (or of its extension) about which the moments of inertia of the body are maximum and minimum.

If coördinate axes are taken to coincide with the principal axes at O, that is with the principal diameters of the momental ellipsoid for O, then the equation of the ellipsoid becomes

$$\frac{X^2}{(Q/k_1)^2} + \frac{Y^2}{(Q/k_2)^2} + \frac{Z^2}{(Q/k_3)^2} = 1$$

where k_1, k_2 and k_3 are the radiuses of gyration of the body with respect to the principal axes at O, for Q/k_1, Q/k_2 and Q/k_3 are the lengths of the semi-diameters (ρ_1, ρ_2 and ρ_3). This equation can be transformed

$$Mk_1^2X^2 + Mk_2^2Y^2 + Mk_3^2Z^2 = Q^2M,$$

or

$$I_1X^2 + I^2Y^2 + I_3Z^2 = Q^2M$$

where I_1, I_2 and I_3 stand for the principal moments of inertia at O. Since there are no ZY, YX, XY terms in this last equation, the products of inertia of the body with respect to coördinate planes as chosen above are zero. Hence, to locate principal axes of a body at any point O, one needs only to find three coördinate planes through the point for each pair of which the product of inertia of the body is zero (see Art. B5); the coördinate axes are the principal axes at the point. And for coördinate axes chosen as described above, equation (1) becomes

$$I = \lambda^2 I_x + \mu^2 I_y + \nu^2 I_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Symmetrical Bodies. — (i) If a homogeneous body is symmetrical with respect to a plane, then any line perpendicular to the plane is a principal axis at the point where it pierces the plane. For, take such line as the x-axis, and the y- and z-axes in the plane, then $\Sigma mzx = \Sigma mxy = 0$, and the equation of the ellipsoid becomes

$$I_xX^2 + I_yY^2 + I_zZ^2 - 2\,DZY = Q^2M$$

which shows that the ellipsoid is symmetrical with respect to the yz plane. Hence the x-axis coincides with one of the axes of the ellipsoid, that is with one of the principal axes of the body at the point O.

(ii) If a homogeneous body has two planes of symmetry at right angles to each other, then their intersection is a principal axis at every point of that line. For if the two planes be taken as coördinate planes and any plane perpendicular to them as the third coördinate plane, then it is obvious that the three products of inertia equal zero; hence the intersection of the planes of symmetry (one of the coördinate axes) is a principal axis at the origin of coördinates (taken at any point on the intersection). If the three products of inertia equal zero, then the ellipsoid is symmetrical with respect to the three coördinate planes, and hence each coördinate axis is a principal axis at the origin. Then if I_1, I_2 and I_3 denote the principal moments of inertia, formula (1) becomes

$$I = \lambda^2 I_1 + \mu^2 I_2 + \nu^2 I_3.$$

B7. Experimental Determination of Moment of Inertia. — When the body is so irregular in shape that the moment of inertia desired cannot be computed easily, then an experimental method may be simpler. There are several experimental methods available.

By Gravity Pendulum. — This method is available if the body can be suspended and oscillated, like a pendulum, about an axis coinciding with or parallel to the line with respect to which the moment of inertia is desired. Let T = the time of one complete (to and fro) oscillation, c = distance from the mass-center to the axis of suspension, W = weight of the pendulum, g = acceleration due to gravity, k = radius of gyration, and I = moment of inertia about the axis of suspension; then

$$k = \frac{T}{2\pi}\sqrt{cg} \quad \text{and} \quad I = \frac{T^2 cW}{4\pi^2}. \qquad \qquad (1)$$

Above formulas are based on the formula for the time of oscillation or period of a pendulum $T = 2\pi\sqrt{k^2/cg}$ (see Art. 184). If the axis of suspension does not coincide with the line about which the moment of inertia is desired, then it remains to "transfer" I to that line (see Art. B3).

The desired moment of inertia can be determined without any time observation as follows: From the same axis about which the suspended body oscillates suspend a "mathematical pendulum," a very small bob with cord suspension (see Art. 184); adjust the length of the cord so that the periods (times of oscillation) of bob and body become equal; then

$$k = \sqrt{cl}, \quad \text{and} \quad I = Wcl/g, \qquad \qquad (2)$$

where l = the distance from the center of the bob to the axis of suspension and k, W, c, I have the same meaning as above. The foregoing result is based on the fact that k^2/c (for the pendulum) equals the length l of the mathematical pendulum (see Art. 184).

By Torsion Pendulum. — The torsion pendulum here referred to would consist of an elastic wire suspended in a vertical position, the lower end being fashioned or terminated in a disk so that objects, whose moments of inertia are to be determined, may be suspended on the wire and made to oscillate about its axis. Let t = the (observed) period (time of one oscillation) of the bare pendulum, t_1 = the (observed) period of the pendulum when it is loaded with a body A which is so regular in shape (as a cube or cylinder) that its moment of inertia about the axis of oscillation can be computed easily, and t_2 = the (observed) period of the pendulum when it is loaded with the body B whose moment of inertia is desired; further let I_1 = the (computed) moment of inertia of A and I_2 = the moment of inertia of B about the axis of suspension. B should be suspended so that the axis of suspension coincides with or is parallel to the line (of B) about which the moment of inertia is desired. Then

$$I_2 = I_1 (t_2 - t) \div (t_1 - t). \quad \ldots \ldots \ldots \quad (3)$$

This result is based on the fact that the square of the period of a torsion pendulum is proportional to the moment of inertia of the pendulum with respect to the axis of oscillation. Thus, if I = the moment of inertia of the bare pendulum, and C the proportionality factor, then

$$t^2 = CI, \quad t_1^2 = C(I + I_1), \quad \text{and} \quad t_2^2 = C(I + I_2).$$

These three equations may be combined so as to eliminate C and I and thus give equation (3).

If B cannot be suspended so as to make the axis of oscillation and the line (of B) about which the moment of inertia of B is desired coincident, then it remains to reduce, or transform, I_2 to that line (Art. B3).

B8. Radius of Gyration of Some Homogeneous Bodies. — Let k = radius of gyration, a subscript with k referring to the axis with respect to which k is taken; thus k_x means radius of gyration with respect to the x-axis. Also M = mass and δ = density.

Straight Slender Rod. — Let l = its length, α = angle between the rod and the axis. Then about an axis through the mass-center $k^2 = \frac{1}{12} l^2 \sin^2 \alpha$; about an axis through one end of the rod $k^2 = \frac{1}{3} l^2 \sin^2 \alpha$.

Slender Rod Bent into a Circular Arc (Fig. B11). — Let r = radius of the arc, then

$$k_x^2 = \tfrac{1}{2} r^2 [1 - (\sin \alpha \cos \alpha)/\alpha], \quad \text{and} \quad k_y^2 = \tfrac{1}{2} r^2 [1 + (\sin \alpha \cos \alpha)/\alpha].$$

The divisor α must be expressed in radians (1 degree = 0.0175 radians). $k_z^2 = r^2$ (the z-axis is through O and perpendicular to the plane of the arc).

Right Parallelopiped (Fig. B12). — The axis OX contains the mass-center, and is parallel to the edge c; $k_x^2 = \frac{1}{12}(a^2 + b^2)$.

Right Circular Cylinder (Fig. B13). — Both axes OX and OY contain the mass-center, r = radius and a = altitude; then

$$k_x^2 = \tfrac{1}{2} r^2; \quad k_y^2 = \tfrac{1}{12}(3r^2 + a^2).$$

Hollow Right Circular Cylinder (Fig. B14). — Let R = outer radius, r = inner radius, and a = altitude; then

$$k_x^2 = \tfrac{1}{2}(R^2 + r^2); \quad k_y^2 = \tfrac{1}{4}(R^2 + r^2 + \tfrac{1}{3}a^2).$$

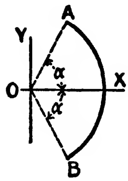

FIG. B11

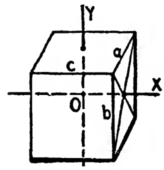

FIG. B12

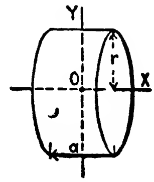

FIG. B13

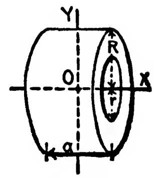

FIG. B14

Right Rectangular Pyramid (Fig. B15). — The x-axis contains the mass-center and is parallel to the edge a; $M = \frac{1}{3}abh\delta$.

$$k_x^2 = \tfrac{1}{12}(b^2 + \tfrac{3}{4}h^2); \quad k_y^2 = \tfrac{1}{20}(a^2 + b^2).$$

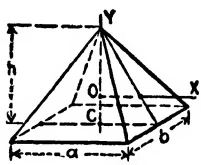

FIG. B15

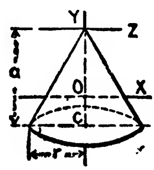

FIG. B16

Right Circular Cone (Fig. B16). — The x-axis contains the mass-center, and is parallel to the base; $M = \frac{1}{3}\pi r^2 a\delta$.

$$k_x^2 = \tfrac{3}{20}(r^2 + \tfrac{1}{4}a^2); \quad k_y^2 = \tfrac{3}{10}r^2; \quad k_z^2 = \tfrac{3}{20}(r^2 + 4a^2).$$

Frustum of a Cone. — Let R = radius of larger base, r = radius of smaller base, and a = altitude. For the axis of the frustum

$$k^2 = \tfrac{3}{10}(R^5 - r^5) \div (R^3 - r^3); \quad I = \tfrac{1}{10}\pi h\delta(R^5 - r^5) \div (R - r).$$

Sphere. — Let r = radius. For a diameter

$$k^2 = \tfrac{2}{5}r^2; \quad I = \tfrac{8}{15}\pi r^5\delta.$$

Hollow Sphere. — Let R = outer and r = inner radius. For a diameter

$$k^2 = \tfrac{2}{5}(R^5 - r^5) \div (R^3 - r^3); \quad I = \tfrac{8}{15}\pi(R^5 - r^5)\delta.$$

Ellipsoid. — Let $2a$, $2b$ and $2c$ = length of axes. For the axis whose length = $2c$,

$$k^2 = \tfrac{1}{5}(a^2 + b^2); \quad I = \tfrac{4}{15}\pi abc\delta(a^2 + b^2).$$

Paraboloid Generated by Revolving a Parabola about its Axis. — Let r = radius of base and h = its height. For the axis of revolution

$$k^2 = \tfrac{1}{3}r^2; \quad I = \tfrac{1}{6}\pi r^4 h\delta.$$

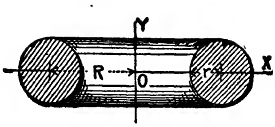

FIG. B17

Ring (Fig. B17). — The x-axis contains the mass-center and is parallel to the plane of the ring; the y-axis is the axis of the ring.

$$k_x^2 = \tfrac{1}{2} R^2 + \tfrac{5}{8} r^2; \quad I_x = \pi^2 R r^2 \delta \, (R^2 + \tfrac{5}{4} r^2).$$

$$k_y^2 = R^2 + \tfrac{3}{4} r^2; \quad I_y = 2\,\pi^2 R r^2 \delta (R^2 + \tfrac{3}{4} r^2).$$

PROBLEMS

CHAPTER II

1. Fig. 1 represents a wood frame to which forces are applied as indicated; the forces are applied by means of ropes tied to nails in the frame. Compute the moment of the 60 lb. force about point 1.

2. A certain chimney is 150 ft. high and weighs 137,500 lbs. Suppose that it is subjected to a horizontal wind pressure of 54,000 lbs., uniformly distributed along its

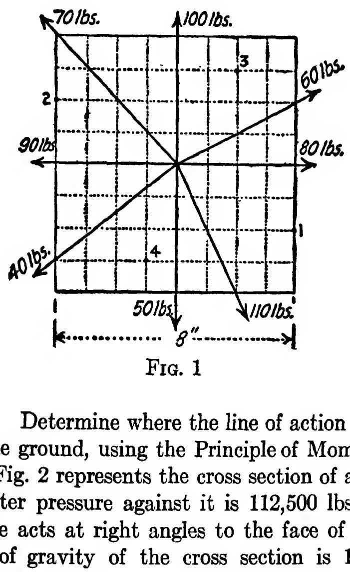

FIG. 1

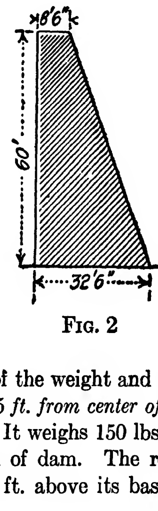

FIG. 2

height. Determine where the line of action of the resultant of the weight and pressure cuts the ground, using the Principle of Moments. *Ans.* 29.45 *ft. from center of base.*

3. Fig. 2 represents the cross section of a masonry dam. It weighs 150 lbs/ft³ and the water pressure against it is 112,500 lbs. per foot length of dam. The resultant pressure acts at right angles to the face of the dam and 20 ft. above its base. The center of gravity of the cross section is 11.46 ft. from the face of the dam and 24 ft. above the base. Find where the resultant of the weight and the pressure cuts the base, using the Principle of Moments.

4. Fig. 3 is a cross section of a "rolling dam." AB is the sheath rigidly fastened to the cylinder which can be rolled upward on two inclined racks CD, one at either end of the dam. The figure shows the dam resting on the bed at A and against the rack at D. In that position the horizontal and vertical components, P_1 and P_2, respectively, of the water pressure are 180 and 30 tons. The weight W is 70 tons. The racks CD are inclined at 55° to the horizontal, and the distance C to D is 9 ft. Without computing the resultant of these three forces, ascertain how far from D its line of action intersects the line CD, using the Principle of Moments.

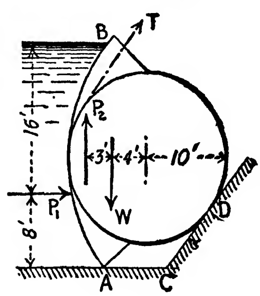

FIG. 3

5. By inspection of Fig. 1, estimate where you might drive the nail and how you would pull the rope to get, from a single force, the same effect as the 50 and 110 lb. forces give Do the same for the 100 and 90 lb. forces. For the 60 and 40 lb. forces,

6. Compound the 80 and 110 lb. forces (Fig. 1) by means of the parallelogram law. Compound also the 50 and 100 lb. forces. (To describe the line of action of the resultant, note where it cuts edges of the square board. Use scales of about 4 ins. and 40 lbs. to the inch.)[1]

7. Compound the 50 and 60 lb. forces (Fig. 1) by means of the triangle law. (Make the vector diagram separate from the space diagram, and use standard notation.)

8. Compound the 60 and 70 lb. forces (Fig. 1) algebraically. (Specify the direction of the resultant by means of the angles between it and the two given forces.)

9. Compound the 50 and 90 lb. forces (Fig. 1).

10. Resolve the 40 lb. force (Fig. 1) into two components, one parallel to the 70 lb. force and one vertical, by a graphical method.

11. Resolve the 100 lb. force (Fig. 1) into two components, one of which acts in the lower edge of the square and the other through the upper right-hand corner.

12. Resolve fully the 60 lb. force (Fig. 1) into two components, one horizontal and one vertical.

13. Resolve fully the 60 lb. force (Fig. 1) into two components, one applied at the upper right-hand corner and one at the lower left-hand corner of the square.

14. Resolve fully the 60 lb. force (Fig. 1) into three components, applied along the two sides and bottom of the frame.

15. Resolve fully the 60 lb. force (Fig. 1) into two components, one of 120 lbs. applied at point 1 and one horizontal.

16. Draw a square and number the corners 1, 2, 3 and 4, consecutively. Imagine a force of 100 lbs. to act in $\overline{12}$ and in the direction $\overline{12}$. Resolve it into components acting in the other three sides.

17. Determine the x and y components of the force system shown in Fig. 1.

18. Compound the 40, 50, 60 and 70 lb. forces (Fig. 1) graphically. (Do not draw the force polygon in the space diagram; use standard notation.)

19. Compound the 70, 90, 100 and 110 lb. forces (Fig. 1) algebraically. (Specify the direction of the resultant by means of the angle between it and the horizontal.)

20. Compound the four forces (wind pressures) represented in Fig. 4. (Be prepared to give the inclination of the resultant and the point where the line of action cuts the floor).

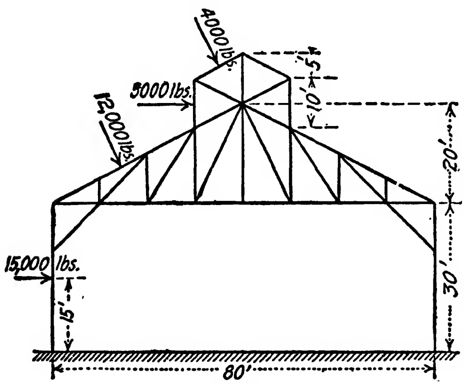

Fig. 4

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[1] In many of these problems it is required that one or more forces be determined wholly or in part. When the complete determination of a force is required, the *magnitude, line of action* and *sense* (or the magnitude, direction, and at least one point in the line of action) must be found and stated. When a problem requires that given forces be compounded, the resultant of these forces must be determined completely; when a problem requires that a given force be resolved into components, these components must be determined completely. In order to make the line of action of a required force seem more real, the student should decide upon an appropriate point of application; for example, a nail or hook in the body to which one could tie a cord or rope for applying the force.

Ans. R = 30,700 *lbs., downward and toward right at 28° to horizontal, cutting floor at 75 ft. from left side of building.*

21. Fig. 5 is a half-section of a building. The four forces are wind pressures, perpendicular to and applied at mid-points of the portions $\overline{01}$, $\overline{12}$, etc. $P = 10$, $P_1 = 7$, $P_2 = 4$, and $P_3 = 2$ tons. Determine completely the resultant of the four forces graphically. (Note where the line of action cuts $\overline{35}$ and $\overline{45}$. The figures in parentheses are coördinates of the points 1, 2 and 3, with respect to O.)

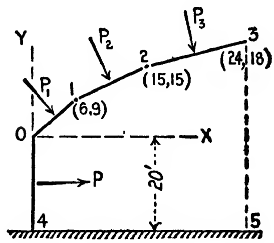

FIG. 5

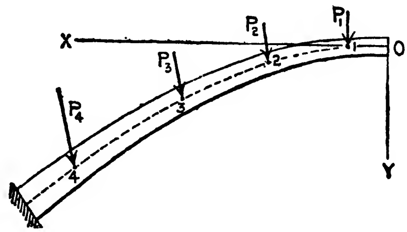

FIG. 6

22. Fig. 6 represents one-half of an arch and certain loads applied to it. $P_1 = 4000$, $P_2 = 5000$, $P_3 = 6000$ and $P_4 = 10{,}000$ lbs.; their inclinations are 0°, 3°, 8° and 12°, respectively; the coördinates of points 1, 2, 3 and 4 are (1.6, 0.1), (4.9, 0.7), (8.4, 2.1), and (12.8, 4.8), all in feet. Compound the four load by the second method. (Specify the line of action of the resultant by means of the angle between it and the x axis and the intercept on that axis.)

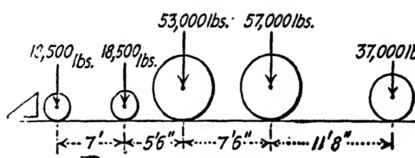

FIG. 7

23. Determine the resultant of the locomotive wheel-loads (Fig. 7).

Ans. R = 184,000 *lbs., vertically downward at* 16 *ft.* 10 *in. to right of first wheel.*

24. On a horizontal line mark off, from left to right and at intervals of 1 inch, points designated as A, B, C, D and E. Consider vertical forces to act through A, B and D as follows: A 10 lbs. up; B 40 lbs. down; D 20 lbs. up. Determine the resultant of the forces at A and D. At A and B.

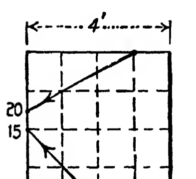

FIG. 8

25. Resolve the 40 lb. force of the preceding problem into components applied at A and E. At C and E.

26. Replace the three forces of Prob. 24 by two forces, one acting through A and the other through E.

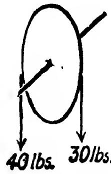

FIG. 9

27. Determine completely the resultant of the four forces described in Fig. 8.

28. Determine the resultant of the loads described in Prob. 22 algebraically.

29. Imagine a clockwise couple of 2 ft-lbs. to act on the square board of Fig. 1. Then compound the couple and the 40 lb. force.

30. Resolve each of the forces shown in Fig. 9 into a force applied at the center of the pulley and a couple. Take diameter of pulley as 3 ft.

31. Solve Prob. 16 by resolving the given force directly into a force and a couple.

32. Solve Prob. 27 as follows: Resolve each of the four given forces into a force acting through the center of the figure and a couple, thus replacing the given system by a set of four concurrent forces and a set of four couples. Then find the resultant of the

four concurrent forces and the resultant of the four couples. Finally, compound the resultant force and the resultant couple, and thus determine the resultant of the original system.

33. Resolve the 30 lb. force (Fig. 10) into x, y and z components.

34. Compound the four forces of the cube in Fig. 10.

Ans. R = 62.6 *lbs., toward right, downward, and backward at angles of* 60.5°, 42.5° *and* 62.5° *to the* x, y *and* z *axes respectively.*

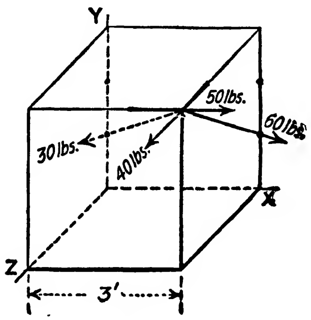

Fig. 10

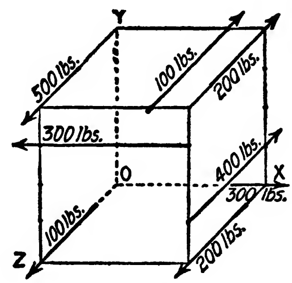

Fig. 11

35. Determine the resultant of all except the 300 lb. forces acting on the 4 ft. cube represented in Fig. 11.

36. A, B, C and D, denote the corners of a horizontal square table top. At each of the corners, A, B and C is applied a downward force of 50 lbs. What fourth force would make the resultant of the four force system pass through D?

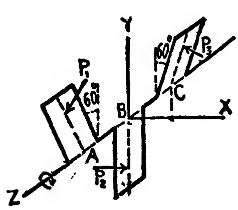

Fig. 12

37. Compute the moments of each of the forces represented in Fig. 10 about the x, y and z axes.

38. Determine the resultant of the three couples acting on the 4 ft. cube represented in Fig. 11. (Specify the plane of the resultant by means of the angles which a normal to the plane makes with the edges of the cube.)

39. Fig. 12 represents a three-throw crank shaft, consecutive cranks being 120° apart. $AB = BC = 9$ in. When the shaft is made to rotate rapidly, as in a balancing machine, the cranks are subjected to equal air pressures. Assume that these are perpendicular to the cranks respectively and have (equal) 4 in. arms; then reduce these three forces to a force applied at B and two couples.

CHAPTER III

40. State what you can about the resultant in the following cases:

(*a*) A system of coplanar concurrent forces for which $\Sigma F_y = 0$; for which $\Sigma M_a = 0$; for which the force polygon closes.

(*b*) A system of noncoplanar concurrent forces for which $\Sigma F_x = 0$.

(*c*) A system of coplanar parallel forces for which $\Sigma F = 0$; for which $\Sigma M_a = 0$; for which the force polygon closes.

(*d*) A system of coplanar nonconcurrent nonparallel forces for which $\Sigma F_x = 0$; for which $\Sigma F_x = \Sigma F_y = 0$; for which $\Sigma M_a = \Sigma M_b = 0$.

41. (*a*) Two couples are in equilibrium; what can you say about them? (*b*) Can a force and a couple be in equilibrium? (*c*) Three parallel forces are in equilibrium; what

can you say about the middle force? (*d*) Is it possible for four forces to be in equilibrium if three are parallel? (*e*) Four forces are in equilibrium and two are known to constitute a couple; what can you say about the other two? (*f*) What can you say about the resultant of a set of coplanar nonconcurrent forces whose force polygon closes?

42. Is it possible to apply three forces along the sides of a triangular board so that they will balance? Is it possible to apply four unequal forces along the sides of a square board so that they will balance?

43. A right circular cylinder of weight W lies in a trough formed by two smooth plane surfaces whose line of intersection is horizontal. The left and right sides of the trough are respectively inclined at angles α and β to the horizontal. Determine the pressures exerted by the sides of the trough against the cylinder for each of the following cases: (*a*) $\alpha = 65°$, $\beta = 40°$; (*b*) $\alpha = 90°$, $\beta = 40°$; (*c*) $\alpha = 115°$, $\beta = 40°$.

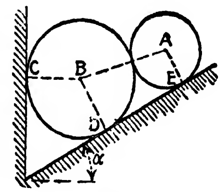

FIG. 13

44. A and B (Fig. 13) are two smooth cylinders supported by two planes as shown. A weighs 200 lbs. and B 100 lbs.; the diameter of A is 6 ft. and of B 10 ft.; $\alpha = 30°$. Determine the pressures on the planes and that between the cylinders.

Ans. Pressure between cylinders = 103.3 *lbs.*

45. The sphere (Fig. 14) weighs 200 lbs.; α, β and γ are respectively 25°, 40° and 30°; all contact surfaces are smooth. Determine the pressures on the wedge M and the tension in the rope.

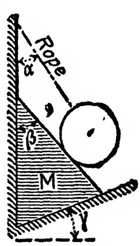

FIG. 14

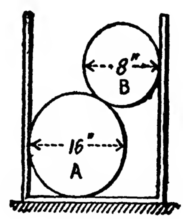

FIG. 15

46. Two right circular cylinders are supported in a box 18 ins. wide as shown in Fig. 15. A weighs 160 lbs., and B weighs 100 lbs. All surfaces are smooth. Find all of the forces acting on each cylinder and properly represent on separate sketches.

47. Fig. 16 represents two wedges; $\alpha = 70°$ and $\beta = 40°$. A push P of 1000 lbs. can sustain what load Q if all rubbing surfaces are smooth?

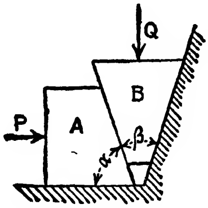

FIG. 16

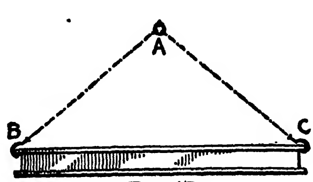

FIG. 17

48. The chains AB and AC (Fig. 17) are 5 ft. long. When BC = 8 ft. and the suspended load W = 2 tons, what is the tension on each chain? If the safe pull for each chain is 3 tons, how large may the spread BC be?

49. Two bars AB and CD (Fig. 18) are connected by a pin at A and to a floor by pins B and C. $BC = 8$ ft., $AB = AC = 5$ ft., and $AD = 8$ ft. A weight of 100 lbs. is suspended from D. Determine the pin pressures at A, B and C.

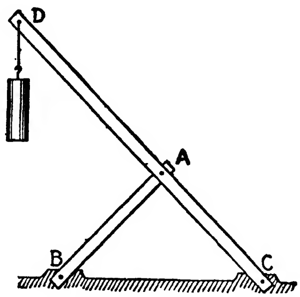

FIG. 18

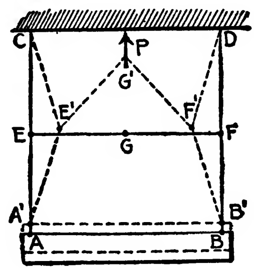

FIG. 19

50. AB (Fig. 19) is a bar suspended from a ceiling by means of vertical ropes AC and BD. The middle points E and F are connected by another rope. $AB = AC = BD = 8$ ft. A vertical force P is applied at the middle G, deflects the ropes as shown by the dotted lines, and raises the bar. How large must P be to support the bar (weighing 1000 lbs.) 6 ins. above its original position? *Ans.* $P = 600$ *lbs.*

51. The cylinder of the steam engine (Fig. 20) is 10 ins. in diameter, the crank AB is 5 ins. long, and the connecting rod BC is 15 ins. long. Assume the engine to be stalled in the position shown, $\theta = 60°$, and the steam pressure 150 lbs/in². Determine the push

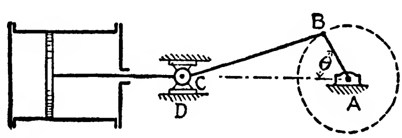

FIG. 20

on the connecting rod BC and the pressure against the cross-head guide D.

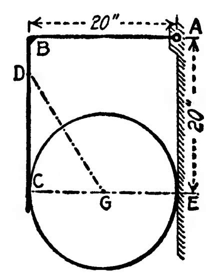

FIG. 21

52. The bell-crank ABC (Fig. 21) is pinned to a wall at A; a cylinder G is suspended by means of a cord from D as shown; $BD = 4$ ins. The cylinder weighs 80 lbs. and is smooth. Determine all the forces which act upon the bell-crank.

53. AB (Fig. 22) is a rigid beam; two hooks are pinned to it at A and B as shown; CD and CE are rods pinned to the hooks and to each other; the hooks engage a heavy body W. $AB = 14$ ft., $CD = CE = 8$ ft., $AD = 6$ in., distance from D to top of body $= 5$ in., $W = 4$ tons. Determine the tension in each rod and all forces acting on one hook. (Neglect weight of parts.)

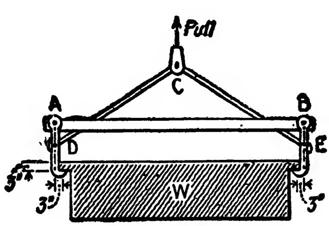

FIG. 22

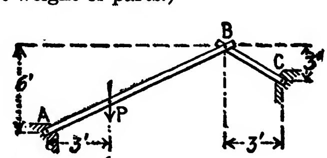

FIG. 23

54. The links in Fig. 23 are pinned at A, B and C; A and C are rigid supports; $P = 100$ lbs. The horizontal distance A to B is 10 ft. Find the reactions at A and C.

55. A homogeneous beam 20 ft. long weighs 50 lbs. At the upper end it rests against a smooth vertical wall and is held at the lower end by a rope 30 ft. long which is also attached to the wall. Find the angle between the beam and the wall, and determine the pull in the rope. *Ans. Angle* = 49.8°.

56. A ring has attached to it three cords A, B and C; all lie in the vertical plane and their directions are respectively up and to the right at 40° to the horizontal, up and to the left at 20° to the horizontal, and down and to the left at 60° to the horizontal. Determine the direction and the magnitude of a pull, which, applied to the ring in the vertical plane, will balance tensions in A, B and C, respectively, of 20, 60 and 90 lbs.

57. Fig. 24 represents a riveting machine operated by compressed air. It consists of a rigid frame F on which the air cylinder C is mounted; P is the piston; AB is the piston rod pinned to the piston at A so that the rod can be rotated somewhat about A inside of the (hollow) piston; the toggle link BD is pinned to the frame at D; the toggle link BE is pinned to the plunger Q (movable in a vertical guide on the frame) at E; HH are the rivet dies between which the rivet is squeezed. AB = 19 ins.; BD = 13 ins.; BE = 10 ins.; the diameter of the cylinder is 10 ins. Assume the air pressure to be 100 lbs/in² and then determine the pressure at the pins D and E, the pressure against the guide, and the pressure on the rivet. (To "lay out" this mechanism begin at D, then fix A, then B, and then E.) Solve the problem when A is advanced 2 ins. from the position shown.

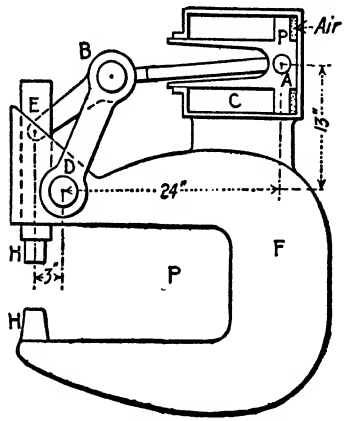

FIG. 24

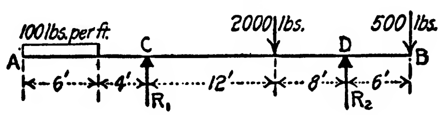

FIG. 25

58. The beam AB (Fig. 25) is supported at C and D, and it sustains three loads as shown. The beam weighs 50 lbs. per lineal foot. Determine each supporting force, or reaction.

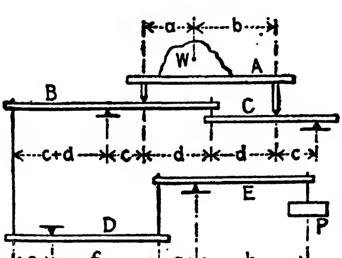

FIG. 26

59. A horizontal beam 20 ft. long is supported by a knife-edge 7 ft. from the left end and is pinned at 6 ft. from the right end. It sustains loads of 5000 lbs. at the left end, 1000 lbs. at 10 ft. from the left end, 2000 lbs. at the right end, and a uniformly distributed load of 1000 lbs. per ft. between the left end and left support. Determine the reactions of the pin and knife-edge.

60. The diagrammatic sketch in Fig. 26 represents a lever system for a scales. Assume W = 250 lbs.; a = 12 ins.; b = 8 ins.; $c = g = 3$ ins.; $d = h$ = 10 ins; $e = 1\frac{1}{2}$ ins.; f = 15 ins. Determine P.

61. A horizontal beam 10 ft. long weighs 60 lbs. per ft. It sustains a concentrated load of 400 lbs. 3 ft. from the left end, is supported at the left end by a vertical rope, and at the right end has a 2 ft. bearing on the horizontal top of a wall. If the tension in the

rope is 510 lbs., what is the total upward pressure of the wall, and where is the center of this pressure?

62. (*a*) *AB* (Fig. 27) is a bar 20 ins. long, and weighs 10 lbs. It rests on a peg *C* and against a smooth wall at *A*, as shown. What vertical force applied at *B* will preserve the equilibrium of the bar?

(*b*) If the weight of the bar is 12 lbs. and a load weighing 4 lbs. is suspended at *B*, at what angle must the bar be placed to insure equilibrium? *Ans.* (*a*) 11.7 *lbs.*

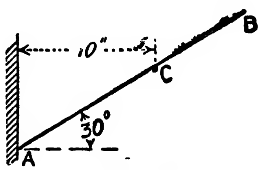

Fig. 27

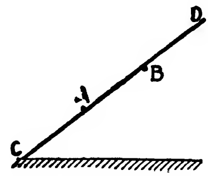

Fig. 28

63. *A* and *B* (Fig. 28) are two horizontal pegs in a wall; they are 3 and 6 ft. above the floor respectively, and the horizontal distance between them is 4 ft. A smooth straight bar *CD*, 15 ft. long and weighing 200 lbs., is placed under *A* and over *B* with its lower end on the floor, but is not sprung into that position. Determine all the pressures on the bar, due to its own weight.

64. A light bar 9 ft. long is fastened to the floor at its lower end by a horizontal pin perpendicular to the bar. The angle between the bar and the floor is 50°; its upper end rests against a smooth vertical wall. Loads of 300 and 200 lbs. are hung at distances of 4 and 7 ft. respectively from the lower end. Calculate the reactions at the ends of the bar.

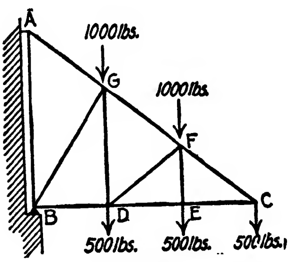

Fig. 29

65. Fig. 29 represents a truss supported by a shelf *B* on a wall and a horizontal tie *A*; *AB* = 9 ft. and *BC* = 12 ft. Determine the reactions at *A* and *B* due to the loads.

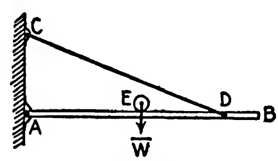

Fig. 30

66. *AB* (Fig. 30) is a beam supported by a rod *CD* and a pin at *A*; *AB* = 9 ft., *AC* = 3 ft., *AD* = 8 ft., and *AE* = 5 ft. The beam weighs 400 lbs. and the load *W* = 1000 lbs. Determine the pull at *C* and the pressure at *A*.

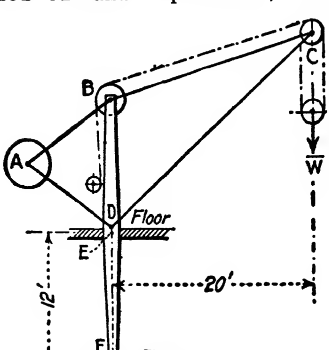

Fig. 31

67. The crane represented in Fig. 31 is supported by two floors as shown. *E* is a hole in the upper floor and *F* is a cylindrical socket in the lower floor. The crane weighs 5 tons and its center of gravity is 2 ft. to the left of the axis of the post. Determine the pressures on the floors when the load *W* is 5000 lbs.

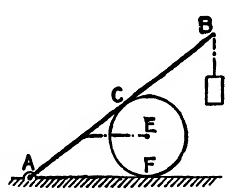

Fig. 32

68. *AB* (Fig. 32) is a bar 12 ft. long fastened to the floor at *A* by a pin and it rests at *C* on a smooth cylin-

der 4 ft. in diameter. The center of the cylinder is 6 ft. to the right of A and is connected by a horizontal cord to the bar at D. A weight of 100 lbs. is hung on the free end of the bar. What is the pressure between the bar and the cylinder; between the cylinder and the floor; what is the tension in the cord; and what is the pressure exerted by the pin on the bar A? Consider the cylinder and the bar as weightless.

Ans. Tension in cord = 120 *lbs.*

69. In Fig. 33 AB = 4 ft., $AC = CD$ = 8 ft.; P = 200 lbs.; and the cylinder E weighs 100 lbs. A, B and E are pin joints. The surfaces at D and F are smooth. The inclination of the plane at F is 30° to the horizontal. Determine the forces acting on the cylinder and those acting on the bell-crank ACD.

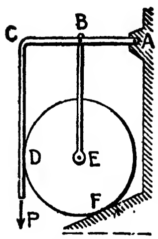

FIG. 33

70. A frame for building and raising a concrete wall is sketched in Fig. 34. The truss B is supported on a trunnion D and by the telescoping piston C of a pneumatic jack F. The dotted lines show the wall and jack in an extreme position. D is 10 ft. vertically and 12 ft. horizontally from E; it is 6 inches from the upper surface IH of the truss; C is 3 ft. from that upper surface; the projections of C and D on IH are 13 ft. apart. The truss weighs 2 tons; its center of gravity is 1 ft. below IH and 8 ft. to the right of D. The load (wall shown) weighs 8 tons; its center of gravity is 7 ins. above IH and 18 ft. to right of D. When the truss is inclined to the horizontal 15° as shown, how large are the pressures at C and D? (Solve graphically.)

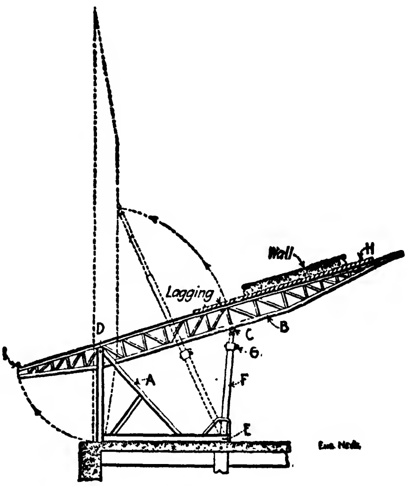

FIG. 34

71. The tripod shown in plan and elevation in Fig. 35 sustains a load of 1000 lbs. Determine the reaction on the lower end of each leg of the tripod.

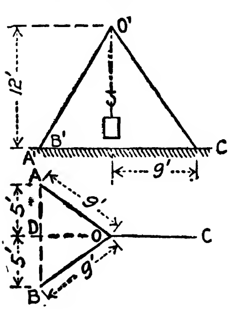

FIG. 35

72. Four spheres of diameter D and weight W are placed in a box whose bottom is 2 D by 2 D. Then a similar sphere is laid upon the four. Calculate the pressures of the spheres upon the sides and bottom of the box.

73. Suppose that the lower spheres (of the preceding problem) are restrained not by the sides of a box but by a ribbon which encircles them. Calculate the tension in the ribbon.

74. Having given necessary dimensions and weight of loading on a table with three legs, how would you proceed to find the reactions or pressures on the lower ends of the legs? Assume simple data and try out your plan. What is your plan for a table with four legs?

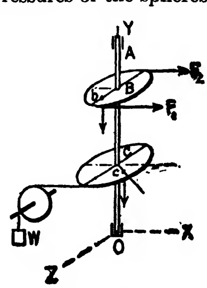

FIG. 36

75. The vertical shaft in Fig. 36 carries a pulley at B weighing 150 lbs. and one at C weighing 80 lbs. The radius of pulley B is 20 ins., of pulley C 15 ins. AB = 2 ft.;

BC = 5 ft.; CO = 4 ft. The center of gravity b of B is 2 ins. from the axis of the shaft; c of C is $\frac{1}{2}$ in. F_1 = 200 lbs.; F_2 = 50 lbs.; and W = 200 lbs. Find the reactions on the bearings at A and O when the angle between the radius through c and the X axis is 30°.

CHAPTER IV

76. The truss represented in Fig. 37 is supported at F and D; $BF = CE$ = 12 ft., $P_1 = P_2$ = 2000 lbs., and $P_3 = P_4$ = 1000 lbs. Determine the amount and kind of stress in each of the members BC, CF and FE.

Ans. Stress in BC = 750 lbs. tension.

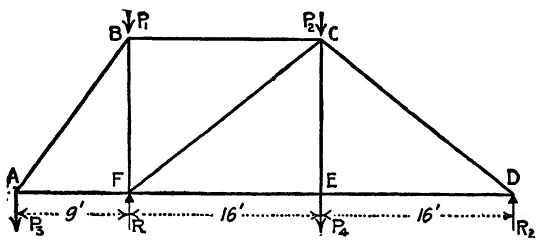

FIG. 37

77. The truss represented in Fig. 38 is supported at A and D; CE = 12 ft., P_1 = 1000 lbs. and P_2 = 2000 lbs. Determine the amount and kind of stress in each member.

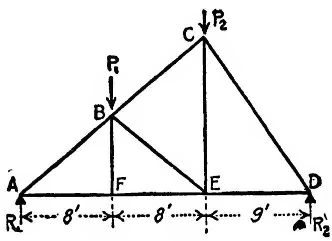

FIG. 38

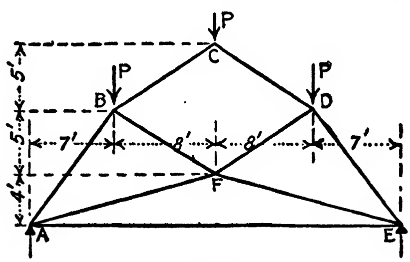

FIG. 39

78. The truss represented in Fig. 39 is supported at A and E; each load P = 1000 lbs. Determine the amount and kind of stress in each member.

Ans. Stress in AB = 1700 lbs. comp.; stress in AE = 1625 lbs. tension.

79. The truss represented in Fig. 40 is supported by a pin at E and by a horizontal tie at A. AE = 9 ft.; $EF = FG = GD$ = 6 ft. Each load P_1 = 1000 lbs.; each load P_2 = 500 lbs. Determine the amount and kind of stress in each member.

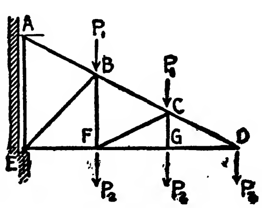

FIG. 40

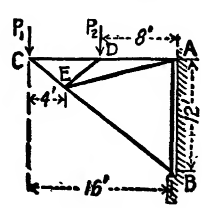

FIG. 41

80. The truss represented in Fig. 41 is held by a pin at joint B and by a horizontal tie at joint A. Each load = 1000 lbs. Solve for the stress in each member.

81. The structure represented in Fig. 42 is a steel head frame for hoisting ore from a mine. The frame is pinned at A and is anchored at B so that either an upward or a downward reaction can occur at that point. The load is 10 tons. The distance $AC = 90$ ft.; the distance $AB = 20$ ft.; the inclined portion of the rope makes an angle of 50° with the horizontal. Determine the amount and kind of stress in each member of the frame.

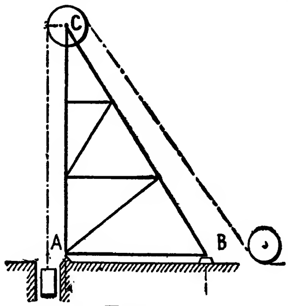

Fig. 42

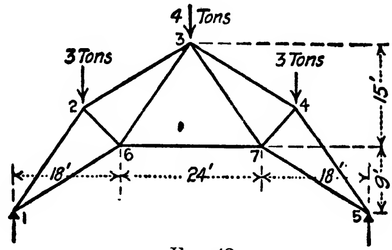

Fig. 43

82. The truss represented in Fig. 43 is supported at each end. The points 1, 2, 3, 6 and the points 3, 4, 5, 7 are at the vertices of parallelograms. Draw a stress diagram for the truss loaded as shown, and make a record of the stresses in the members.

83. The truss represented in Fig. 44 is supported at each end. The total width of the truss is 32 ft.; the total height is 20 ft.; the horizontal members are 8 ft. long; the vertical member is 10 ft. long. Each load $P_1 = 1000$ lbs., and $P_2 = 2000$ lbs. Determine the amount and kind of stress in each member graphically.

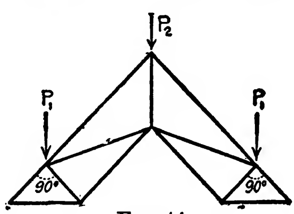

Fig. 44

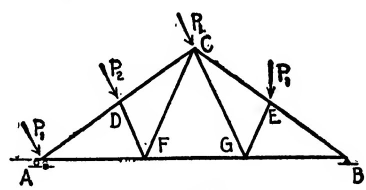

Fig. 45

84. The truss represented in Fig. 45 is held by a pin at the right end and rests upon a smooth support at the left end. The span is 30 ft.; the height is 12 ft. Each load $P_1 = 1000$ lbs.; $P_2 = 2000$ lbs. Solve graphically for the stress in each member.

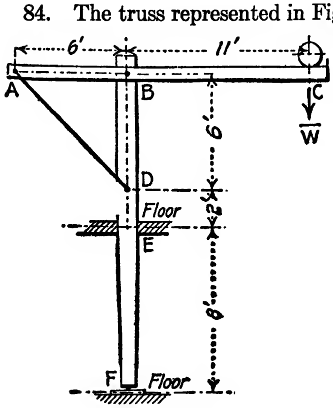

Fig. 46

85. Fig. 46 represents a crane consisting of three members, a boom AC, a brace AD, and a post BF. The crane is supported at E and F by two floors. The load $W = 5$ tons. Determine all forces acting on each member.

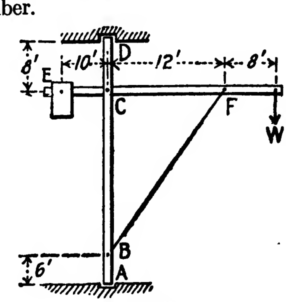

Fig. 47

86. The crane represented in Fig. 47 rests in a socket at A and bears against the smooth side of the hole in the floor at D. There are pins at B, C and F. The load W

is 4000 lbs., the counterweight E weighs 5000 lbs. BC = 14 ft. Determine all the forces which act upon each member of the crane.

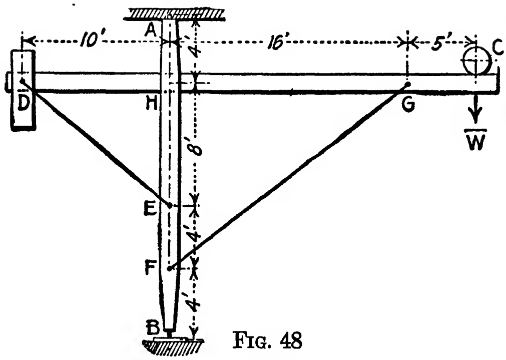

Fig. 48

87. The crane represented in Fig. 48 consists of a post AB, a boom CD, and braces DE and FG. The crane is supported by sockets at A and B as shown. The boom passes freely through a smooth slot in the post at H so that any reaction existing there will be vertical. The counterweight at D is $\frac{1}{2}$ ton, the load W is $\frac{1}{2}$ ton, and the latter is 21 ft. from the axis of the post. Determine all the forces which act upon each member. *Ans. Force at H on boom* = 1.1 *tons down.*

88. Fig. 49 represents a certain type of hydraulic crane. It consists of a post AB, an hydraulic cylinder C mounted on the post, a large sleeve S which can be slipped along the post, two rollers D and E mounted on the sleeve, a boom EF, and a tie rod FG. When water (under pressure) is admitted to the cylinder, the pistons are pushed upward; the upper one bears against the sleeve, and rolls the entire part $DEFG$ up along the post. Let the load W = 2 tons and suppose that it is 10 ft. out from the axis of the post; then determine all the forces which act upon each pin (D, E and G).

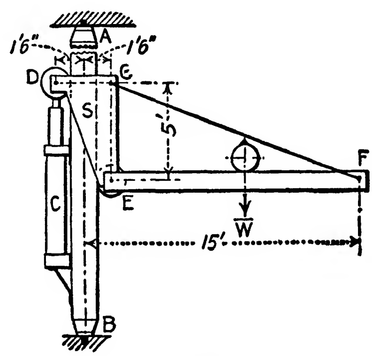

Fig. 49

89. Solve Prob. 85 but take into account the weights of the members which are as follows: post BF = 0.5 ton, brace AD = 0.2 ton, and boom AC = 0.7 ton. The boom is 18 ft. long; its center of gravity is 2 ft. 6 ins. from B.

90. The crane shown in Fig. 50 rests in a socket at A and passes through a hole in the floor at B, the sides of the hole affording horizontal support. The diameter of each pulley is 2 ft. ED = 8 ft. The rope is vertical from F to D and is fastened to pulley on the post at D. The load W is 8 tons. Determine all forces acting on each member.

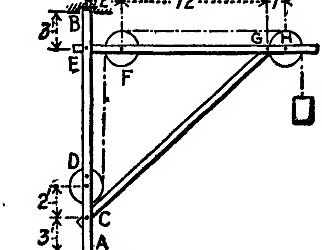

Fig. 50

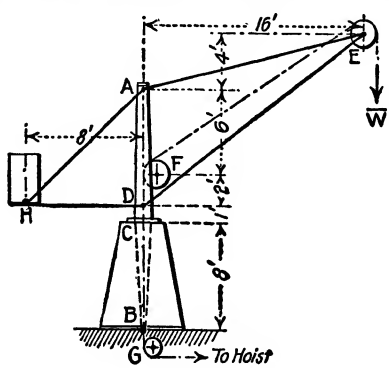

Fig. 51

91. Fig. 51 represents a crane supported by a foot-step bearing at B and a collar-

bearing at C. B can furnish horizontal and vertical support, and C can furnish horizontal support only. The pulleys E and F are 1 ft. in diameter; the hoisting cable enters the post at F, descends through the post, over pulley G, and to the hoist as shown. The counterweight H is 2 tons and the load 4 tons. Determine all the forces which act upon each member. *Ans. Force on post at* $C = 6.25$ *tons.*

92. The crane represented in Fig. 52 consists of a post AB, a boom CD, and a tie rod DE. The pulley at D and the winding drum at G are 1 ft. in diameter. The load W is 1 ton. $DE = 12$ ft. Determine all the forces which act on each member.

93. Imagine the winding drum (Prob. 92) to be mounted in bearings at H (supported by the brace CD) instead of at G. Then solve.

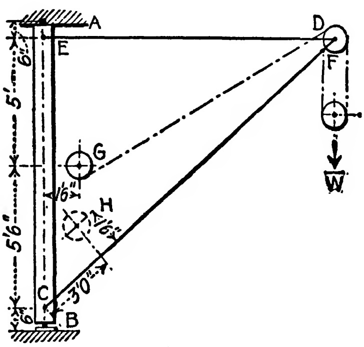

FIG. 52

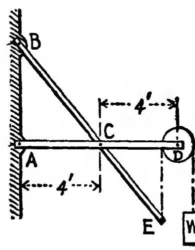

FIG. 53

94. In Fig. 53 the members are pinned to the wall at A and B, and to each other at C. The diameter of the pulley is 2 ft.; the load W is 1000 lbs.; $AB = 5$ ft. Determine the forces which act upon each member of the structure.

CHAPTER V

95. A (Fig. 54) weighs 100 lbs. and B 200 lbs. A, B and C are very rough. Make separate sketches of A and B and represent all the forces which act on each body when $P = 20$ lbs. (not large enough to produce any slipping).

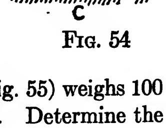

FIG. 54

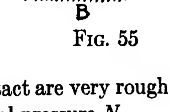

FIG. 55

96. A (Fig. 55) weighs 100 lbs.; the surfaces in contact are very rough; $P = 50$ lbs., and $\alpha = 20°$. Determine the friction F and the normal pressure N.

97. Same as Prob. 96 but refer to Fig. 56.

98. A (Fig. 55) weighs 100 lbs., $\alpha = 40°$, $\mu = 0.6$, $P = 200$ lbs. Does P move A?

99. Same as Prob. 98 but refer to Fig. 56.

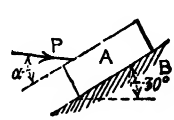

FIG. 56

100. Two horizontal pegs A and B project from a wall; B is 2 ft. above and 3 ft. to the right of A. Can a uniform slender bar 10 ft. long be placed over B and under A in such a way as not to slip down, if the coefficient of friction between the bar and the pegs is 0.3?

101. Can a cylinder be rolled over a cleat the height of which is 0.2 times the diameter of the cylinder, by a force applied at the top of the cylinder, if the coefficient of friction between cleat and cylinder is 0.4?

102. A (Fig. 57) weighs 100 lbs. and B 200 lbs. For A and B, $\mu = \frac{1}{2}$; for B and C, $\mu = \frac{1}{3}$. How large must P be to cause slipping? *Ans. More than* 150 *lbs.*

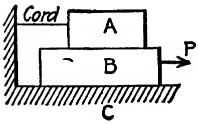

FIG. 57

103. A (Fig. 55) weighs 100 lbs.; $\alpha = 20°$, and $\mu = 0.6$. How large must P be to start A? How large is F when slipping impends?

104. A straight bar rests in a vertical plane with one end on a horizontal floor and the other against a vertical wall. The coefficient of friction for floor and bar is 0.4; for wall and bar, 0.3. At what minimum angle between bar and floor would the bar rest?

105. Suppose that the bar of the preceding problem weighs 100 lbs., and is set at an angle of 60°. Determine the necessary downward force applied at the upper end to cause slip of the bar. At what angle would the bar have to be placed in order that slipping could not be made to occur in this way?

106. The ladder AB (Fig. 58) is 40 ft. long and weighs 100 lbs. The coefficient of friction at A (between pole and rung of ladder) is $\frac{1}{5}$; at B $\frac{1}{4}$. A is 25 ft. above the ground. Compute the force P required to overcome gravity and friction in the position shown.

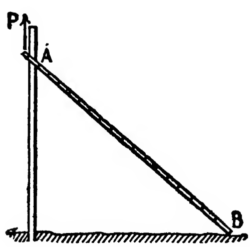

FIG. 58

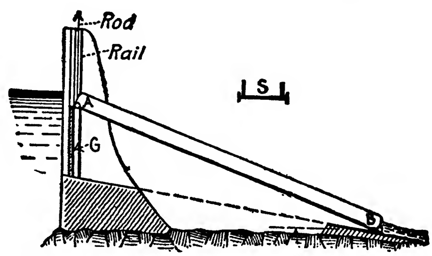

FIG. 59

107. Figure 59 represents the cross section of a dam, a sluice gate, and a log sluice or trough AB (shown in section at S). Water is shown passing over the gate and down the sluice permitting the passage of logs. The sluice is made adjustable to the water level. The front wheels at A rest against vertical rails, and the wheels at B on rails inclined at 10° to the horizontal. The end A is raised or lowered by means of a vertical rod operated from above by a suitable winch. The log sluice weighs 10 tons, AB = 80 ft. Determine the pull at the rod required to overcome the weight of the sluice and the friction at A and B when the sluice is inclined at 25° to the horizontal. (The diameter of the wheels A and B is small compared to AB; so regard the sluice as slipping on the two ends (like the ladder in the preceding problem) and take the equivalent coefficient of friction as $\frac{1}{10}$.)

108. The coefficients of friction between A and B and B and C (Fig. 60) are $\frac{1}{4}$. The top of B is inclined at 20° to the horizontal. W weighs 10 tons. How great must P be to start the wedge B?

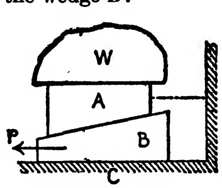

FIG. 60

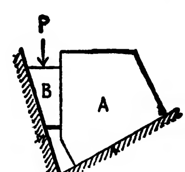

FIG. 61

109. A (Fig. 61) weighs 10,000 lbs.; the coefficient of friction for all contacts is $\frac{1}{2}$. The surface against which B rests is inclined at 70° to the horizontal, that against which A rests at 30°. What value of P is required for starting A up the plane?

110. Fig. 62 represents a double-wedge device for raising and lowering a heavy load W. The device consists of wedges A and B and bearing blocks C and D; $W = 200{,}000$ lbs. The coefficient of friction is 0.5. How large are the required pushes P to raise the load? How large are the required pulls to lower the load? (First consider C and determine the forces acting upon it.)

Ans. $P = 171{,}000$ *to raise the load.*

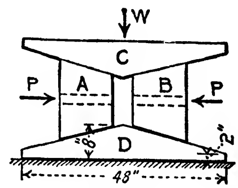

FIG. 62

111. Fig. 63 represents, somewhat conventionalized, an adjusting device used in making the closure (insertion of the last few members) of a large cantilever bridge (Beaver River). The mechanical elements are a double wedge W, a screw S, and a lever L. The accessories are a head piece H, two struts A, and two wedge-blocks B; they are pin-connected as shown. C and C are two portions of the bridge member to be connected; they are under compression P and pin-bear against the wedge blocks B. The nut, which bears against the head piece, can be turned by means of the lever, and the screw and wedge raised or lowered. Raising the wedge separates the wedge blocks and parts C and C. Determine the necessary moment (of force) on the lever for raising the wedge against pressures $P = 1{,}235{,}000$ lbs., assuming that the struts A are vertical and the following data: mean diameter of screw = $4\frac{1}{2}$ ins.; pitch of screw = $\frac{1}{4}$ in.; bevel of wedge (each side) = 1 in 10; mean radius of nut where it bears on the head piece = 9 ins.; coefficient of friction for all rubbing surfaces = $\frac{1}{4}$. (Consider first a wedge-block, and determine all the forces which act upon it.)

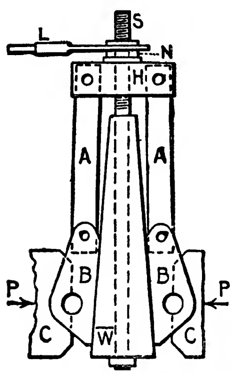

FIG. 63

112. Fig. 64 represents a band-brake. The diameter of the wheel is 1 ft. 8 ins., the angle of lap = 255°, $P = 60$ lbs., and the coefficient of friction is $\frac{1}{5}$; the wheel is turning clockwise. Compute the frictional moment and the pull on the pins A and B. Solve for the case when the wheel is turning in the other direction.

Ans. *Moment* = 3270 *ft-lbs. when turning clockwise.*

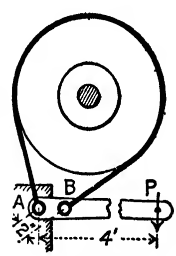

FIG. 64

CHAPTER VI

113. It was desired partially to locate the center of gravity of a heavy irregular body on a truck. The loaded truck was run up to a street scales and the load on the front pair of wheels weighed and then that on the rear; they were 4000 and 8000 lbs. respectively. The wheel loads on the unloaded truck were found likewise to be 2500 and 3500 lbs. respectively. The wheel base was 132 in. What information about the center of gravity of the heavy body is afforded by these data?

114. In order partially to locate the center of gravity of an elongated irregular body, it was supported at two points A and B at same level, resting on a scale first at A and then at B. The weights recorded were 1450 lbs. and 785 lbs. respectively; the distance $AB = 3$ ft. What information about the position of the center of gravity of the body is afforded by these data?

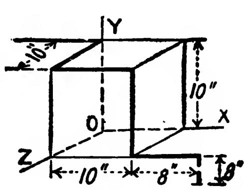

FIG. 65

115. Fig. 65 represents a bent wire; the length of the part extending to the left of the y axis is 12 in. Determine the coördinates of its center of gravity.

Ans. $\bar{x} = 5.76$, $\bar{y} = 5.83$, $\bar{z} = 7.07$ *in.*

116. Fig. 66 represents a thin plate into which holes were punched at A and B, and the pieces glued on at C and D, respectively. Area of hole A = 4 in.^{2}; that of B = 2 ins.^{2}. Locate the center of gravity of the modified plate.

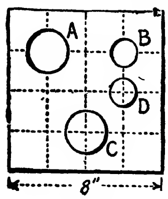

FIG. 66

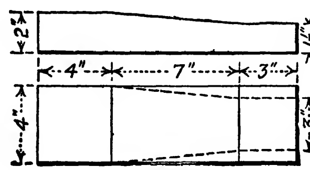

FIG. 67

117. Fig. 67 represents a crank-arm for a shaft, by plan and elevation — dotted lines to be disregarded. Locate the center of gravity of the arm.

118. Solve Prob. 117 but change width at thin end as shown by dotted lines. (See Obelisk, Art. 94.)

119. Fig. 68 is the cross section of a steel beam "built-up" of two angles $5 \times 4 \times \frac{1}{2}$ in. and a plate $8 \times \frac{3}{4}$ in. The centroid of each angle is 1.57 ins. from the back of the shorter leg. Determine the position of the centroid of the entire section.

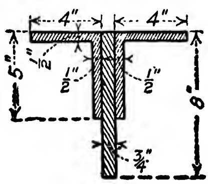

FIG. 68

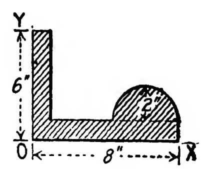

FIG. 69

120. Determine the coördinates of the centroid of the shaded area in Fig. 69. The width of each rectangle is 1 in. (See Art. 94 for centroid of semicircular area.)

121. Determine the centroid of the shaded area in Fig. 70. The height of the rectangle is 10 ins.; of the triangle, 9 ins. The area of the hole is 8 sq. ins. and the coördinates of its centroid are $x = 3$ and $y = 4$ ins.

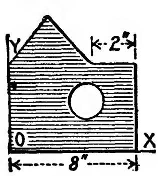

FIG. 70

122. Prove that the distance of the centroid of a triangle from its base equals one-third the altitude.

123. Prove the statements on page 113 about the centroid of a right circular cylinder.

124. Imagine an ellipsoid separated into halves by one of its planes of symmetry. Determine the position of the centroid of one of these halves.

125. Prove that the distance from the centroid to the base of a paraboloid of revolution formed by revolving a parabola about its axis equals one-third the altitude a (see Fig. 186, page 109).

CHAPTER VII

126. A cord is supported at two points on the same level 30 ft. apart, and its lowest point is 8 ft. below the level of the supports. If the load is 20 lbs. per horizontal ft., what are the tensions at the supports and at the lowest point?

Ans. 411 *lbs. at supports,* 281 *lbs. at lowest point.*

127. A certain wire weighs 0.094 lbs/ft. and can sustain a pull of 1500 lbs. with safety. It is to be suspended between two points on the same level and 1000 ft. apart. Assume that the suspended wire will be parabolic, and compute the shortest piece of wire that may be used.

128. Solve the preceding problem but on the assumption that the suspended wire takes the catenary form.

129. A cable is to be suspended between two points at the same level 200 ft. apart; the sag is to be 80 ft. Determine the length of the cable.

130. A rope 100 ft. long is suspended from two points A and B at the same level 80 ft. apart. A body weighing 1000 lbs. is suspended from a point C, x ft. distant from A. Determine the tension in AC when $x = 20, 30, 40, 50, 60, 70$ and 80 ft. Make a graph showing how the tension varies with x.

131. A rope is to be suspended at its two ends from two points AB on the same level and 40 ft. apart. Heavy weights are to be hung from knots on the rope so that the rope will assume the form of half an octagon. What weights will hold the rope to the desired form?

CHAPTER VIII

132. Reduce a sprint of 100 yards in 10 seconds to miles per hour. Express a velocity of 60 miles per hour in feet per second. Express a velocity of 100 kilometers per hour in feet per second.

133. A point P moves in a straight line so that $s = 2\,t^3 - 5\,t^2$, where s (in feet) equals the distance of P from a fixed origin in the path at any time t (in minutes). Determine the velocity when $t = 1$ min.; when $t = 2$ mins. Interpret the negative sign.

134. A point moves in a straight line in accordance with the law $s = t^3 - 40\,t$, where s is distance in feet from a given point in the path and t is time in seconds. By calculus find the velocity when $t = 5$ sec. What is the average velocity for the second preceding the instant named? For the second following the instant?

135. A certain point P of a mechanism is made to move in a straight line by means of a crank in such a way that $s = 3 \cos 2\,\theta$, where s = the (varying) distance in feet of P from a fixed origin in the path of P, and θ = the (varying) angle which the crank makes with a fixed line of reference. The crank rotates uniformly at 100 rev/min. Determine position and velocity of P when $\theta = 60°$. Interpret signs of the results.

Ans. Velocity = $-$ 54.4 *ft/sec.*

136. In a certain "gunnery experiment" the shot was fired through screens placed 150 ft. apart. The times (in seconds) of piercing were observed with the following results:

screen	1	2	3	4	5	6	7
time	0	0.0666	0.1343	0.2031	0.2729	0.3439	0.4161

Determine the velocity at the fourth screen.

137. Fig. 71 is a chronographic record of the launching of the U.S.S. California (*Transactions of Naval Architects and Marine Engineers*, Vol. 12). Determine the velocity of the ship at the twentieth second in the following three ways: first, from the average velocities for at least four intervals after the instant; second, from the average velocities for at least four intervals before the instant; third, from the average velocities for the half-seconds immediately before and after the instant.

138. Compare the retardation of a train at 4 miles per hour per second with the retardation of gravity on a ball thrown vertically upward. A certain electric car can get up a speed of 60 miles per hour in 20 seconds. Compare its average starting acceleration with the acceleration of gravity.

139. In a certain rectilinear motion $v = t^2 - 10\,t$, where v is velocity in feet per minute and t is time in minutes. Determine by calculus the acceleration when $t = 10$ min. What is the average acceleration for the minute preceding the instant named? For the minute following the instant?

140. For the motion described in Prob. 133, determine the acceleration when t = 1 min.; when $t = 2$ mins. Interpret the signs of the results.

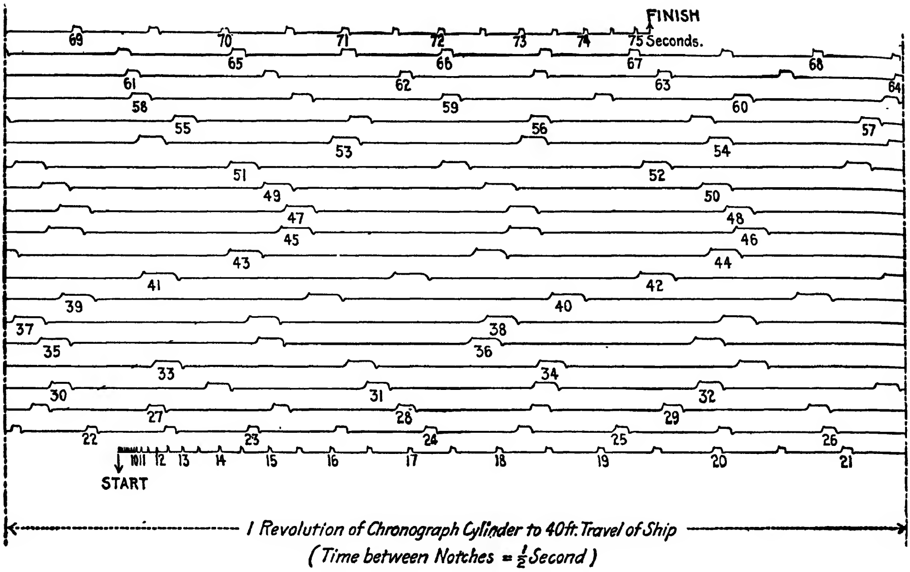

FIG. 71

141. For the motion described in Prob. 135, determine the acceleration of P when $\theta = 60°$. Interpret the sign of the result. *Ans. Acceleration* = 658 *ft/sec/sec.*

142. A point moves in a straight line in accordance with the law $s = 2 \sin (0.05\,t + \pi)$, where s is in inches, t in seconds, and the angle in radians. Determine the velocity and acceleration when $t = 0$; when $t = 20$. Interpret the signs of your results.

143. "This problem was encountered in our studies of hydraulic drives for locomotives:

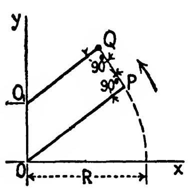

FIG. 72

The point Q (Fig. 72) represents the center of gravity of a certain mass which moves in the following manner. The line OP, of a constant length R, rotates around the fixed center O with a constant angular velocity. From another fixed point O_1 the line O_1Q is drawn parallel with OP. The line PQ is perpendicular to OP. Determine the maximum accelerations of the point Q in the directions O_1Q and PQ." (Westinghouse Electric and Manufacturing Co., Problem Service for Technical Schools.)

144. In a certain run the velocity of an electric car changed in the following manner:

Time t,	0	10	15	20	25	30	35	40	50	114	125 sec.
Velocity v,	0	15	21	24.5	26.5	28.5	29.5	31	32.5	21	0 mi/hr.

The car coasted during the interval 50–114 and was braked during the interval 114–125. (Sheldon and Hausman, *Electric Traction and Transmission Engineering*, p. 63.) Find the acceleration when $t = 0$ sec.; when $t = 25$ sec.

145. The following velocities (feet per second) were computed from the chronographic record (Fig. 71) by taking the mean of the average velocities for the half-seconds immediately preceding and following the instants or times listed below.

$t =$	15	16	17	18	19	20	21	22	23	24
$v =$	2.50	3.00	3.55	4.20	4.80	5.45	6.10	6.75	7.45	8.15

Compute the acceleration for $t = 16$ secs.

146. A point P moves in a straight line so that $a = 4 - 2t$, where a is in feet per minute per minute and t in minutes. When $t = 0$, $v = 0$ and $s = 0$. Determine general formulas for v and s. What are v and s when $t = 4$? When $t = 5$?

147. A body has an "initial velocity" (when $t = 0$) of 15 ft/sec. and an acceleration expressed by $a = 90t - 24t^2$, the units being foot and second. (*a*) How far does the body move in the interval $t = 3$ to $t = 7$? (*b*) What are the velocities at the instants named? (*c*) What is the average velocity for the interval?

148. Draw the space-time curve for the motion of the projectile of Prob. 136 and determine the velocity at the fourth screen from the graph.

149. Draw the distance-time graph for the interval from 15 to 24 secs. of the launching mentioned in Prob. 137, and determine the velocity at the twentieth second from the graph.

150. Draw the velocity-time graph for the data in Prob. 144. Determine the acceleration when $t = 25$ sec. What is distance covered from $t = 25$ to $t = 75$ sec.?

151. Draw the velocity-time graph for the interval from 15 to 24 secs. of the launching mentioned in Prob. 137, and determine the acceleration at the twentieth second from the graph.

152. Make a sketch of the velocity-time graph for the train-run described in Prob. 160, calling the lengths of the three periods t_1, t_2 and t_3 respectively. Then use the principle that "area under the curve" represents distance travelled to find values of t_1, t_2 and t_3, and finally the time for the entire run.

153. Fig. 73 shows the acceleration-time graph for a certain rectilinear motion. When $t = 0$, v and $s = 0$. Construct the v-t and s-t graphs.

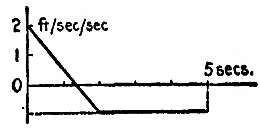

FIG. 73

154. A point moves so that the a-t curve is a straight line passing through the origin and sloping upward to the right. When $t = 0$, $s = 0$ and $v = 0$. Show by sketches the general form of the v-t curve and the s-t curve.

155. Explain and illustrate the difference that would be made in each of the curves of Prob. 154 by each of the following changes in initial conditions, considered singly: (*a*) An initial negative acceleration; (*b*) an initial positive acceleration; (*c*) an initial negative velocity; (*d*) an initial positive velocity; (*e*) an initial negative displacement; (*f*) an initial positive displacement.

156. If a portion of the s-t curve for a given motion is concave downward, what can you say of the corresponding part of the v-t curve? Of the a-t curve? What if the s-t curve is concave upward? What if it is straight? What does a point of inflection on the s-t curve indicate with respect to velocity? With respect to acceleration? Is a v-t curve completely determined by the corresponding s-t curve? Is an s-t curve completely determined by the corresponding v-t curve? Explain.

157. In testing automatic safety cushions placed at the bottom of elevator shafts in the Woolworth Building, N. Y., the following data were obtained:

Distance from top of air cushion,	0	20	40	60	80	100	120	130	135	137 ft.
Pressure on bottom of car,	4	4	7	10	12	9	9	9	8	0 lb/in.^{2},
Downward velocity of elevator,	168	168	157	140	116	92	67	52	32	0 ft/sec.

Plot the distance-velocity curve, and find the acceleration when the elevator had fallen 70 ft. in the cushion. (*Engineering Record* for Sept. 5, 1914.)

158. From the data in the preceding problem find the time consumed in falling 70 ft. in the air cushion. (Hint. Plot the reciprocals of the velocities against the corresponding distances.)

159. A certain train can be retarded at a rate of 4 mi/hr/sec. by braking. Deter-

mine the times (in seconds) and the distances (in feet) in which the train can be stopped from 10, 20, 30 and 40 mi/hr. (Assume that the retardation is the same at all speeds.)

Ans. *From* 40 *mi/hr. in* 10 *secs. and in* 293.3 *ft.*

160. A certain electric train can get up full speed of 24 mi/hr. in a distance of 150 ft. and can stop from full speed in a distance of 100 ft. What is the shortest time in minutes in which the train can make a run between two stations 650 ft. apart, the train starting from one station and coming to full stop at the other? (Assume that the starting and stopping are accomplished uniformly with respect to time.)

161. In a certain series of tests on emergency stops with an automobile the following data were obtained:

Speed of auto,	6	10	15	20	25	30	mi/hr.
Stopping distance,	1.67	6.00	9.50	36.8	42.0	47.5 ft.	

(*Engineering News* for Sept. 7 and Oct. 10, 1912). Compare the average retardations.

162. A train can get up a speed of 60 mi/hr in 5 min., and stop in 0.5 mi. About midway between two stations 10 mi. apart a bad piece of track one mile long necessitates reduction of speed to 10 mi/hr. Assuming that acceleration and retardation can be applied uniformly with respect to time, determine the time between stations. (Sketch the velocity-time graph before calculating.)

How much time was lost on account of defective track?

163. A point Q describes a simple harmonic motion; the frequency = 100 (to and fro) oscillations per minute and the amplitude = 3 ft. Determine the average accelerations for Q for the following distances traversed: first 6 ins. from one end of its path; second 6 ins.; third 6 ins.; and first 18 ins. *Ans.* *For first* 6 *ins.*, 310 *ft/sec/sec.*

164. The period of a certain simple harmonic motion is 8 secs., and the amplitude is 6 ins. What is the maximum velocity? The maximum acceleration? For the motion from one extreme point in the path to the center, what is the average velocity? The average acceleration?

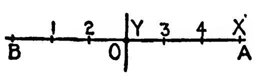

FIG. 74

165. Four particles, Q_1, Q_2, Q_3 and Q_4, are describing simple harmonic motions in AB (Fig. 74); the period of each motion is 8 secs. At a certain instant the four particles are at points 1, 2, 3 and 4, respectively; Q_1 and Q_3 are moving toward the right and Q_2 and Q_4 are moving toward the left. Write out the expressions for the x coördinates of the moving points t secs. after the instant mentioned. (AB = 12 ins., and is divided into sixths by the points.)

166. Fig. 75 represents, in principle, a certain "throw" testing machine for subjecting a metal specimen to rapid changes of direct stress (tension and compression). S is the specimen, firmly screwed into two bosses M and N. W is a weight firmly fastened to the lower boss. The parts named can be oscillated in the vertical guides G by means of an ordinary crank-connecting rod mechanism (OP-PC). When the machine is not running, the specimen is subjected to a tension equal to the weight of N and W. When the machine is running, the stress on the specimen changes continuously, its value at any instant depends on the acceleration of the weight. Let $OP = \frac{1}{2}$ in., PC = 9 ins., weight of N and W = 25 lbs., and speed of crank = 2000 rev/min. Determine the acceleration of the weight W at each end of a stroke or oscillation, and at the middle of the stroke.

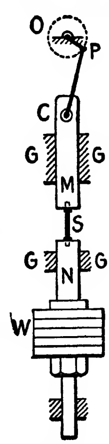

FIG. 75

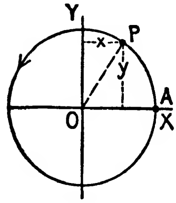

FIG. 76

167. A point P starts at A (Fig. 76) and moves in the circle as indicated, traversing distance s so that $s = 2t^2$, where t is time after starting in seconds and s is in feet; radius

$OA = 20$ ft. Determine the velocity of P when $t = 3$ and when P has traveled 40 ft. *Ans. When $t = 3$, $V = 12$ ft/sec. up and to left at 38.4° to horizontal.*

168. For the motion of P, Prob. 167, determine the average acceleration for the first 3 seconds. For the interval between $t = 3$ and $t = 5$.

169. For the motion of P, Prob. 167, draw the hodograph for the first 3 secs. Then determine the average accelerations for the intervals 1 to 3, 1.5 to 3, 2 to 3, 2.5 to 3. Next determine the magnitude and direction of the acceleration when $t = 3$ from these average accelerations.

Ans. $A = 8.2$ ft/sec/sec. to left and down at 22.5° to horizontal.

170. For the motion of P, Prob. 167, derive expressions for v_x and v_y, and determine the velocity by means of its x and y components when $t = 3$.

171. The point Q (Fig. 77) on the rim of a wheel rolling in a straight line describes a curve known as cycloid. Let v' = velocity of the center of the wheel C, a' = the acceleration of C, and R = radius of the wheel. Find formulas for the x and y components of the velocity of Q when in the position shown. (Let s = the abscissa of C, and x and y = the coördinates of Q. Then $x = s - R \sin \theta$, and $y = R(1 - \cos \theta)$; also $s = R\theta$.)

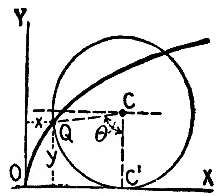

FIG. 77

172. For the motion of P, Prob. 167, derive expressions for a_x and a_y, and determine the acceleration by means of its x and y components when $t = 3$.

173. For the motion of Q, Prob. 171, derive expressions for a_x and a_y.

174. With reference to an origin O, a vertical axis OY, and a horizontal axis OX, a point starts from rest on the X axis and moves according to the equations: $x = 10 + 4t^2$ and $a_y = 3t$, the units being the foot and second.

(*a*) Determine the position, velocity and acceleration of the point for the instant $t = 3$. (*b*) Determine the average acceleration of the point for the interval $t = 0$ to $t = 3$. Represent the results of your solution on a sketch.

175. For the motion of P, Prob. 167, derive expressions for a_t and a_n, and determine the acceleration by means of its tangential and normal components when $t = 3$.

176. A particle is moving at a constant speed of 2 ins/sec. along an ellipse whose axes are 6 and 2 ins. long. Determine the amount and direction of the acceleration of the particle at the instant it is passing one end of the longer axis.

177. Show the approximate form of the path followed by the point of Prob. 174 for a short distance on either side of the position occupied by the point when $t = 3$. Determine the radius of curvature of the path there and the rate at which the speed of the point is changing at that instant.

178. Does the acceleration of a point moving with uniform speed along a curved path depend on the direction in which the point is moving? Upon what does it depend? Upon what does it depend if the point is moving with varying speed? Could a point move along a curved path in such a way as to have zero acceleration? What are the limits as to direction of acceleration in curvilinear motion? Could a point on a curved path have, at a given instant, acceleration but no velocity?

CHAPTER IX

179. Solve Prob. 171, regarding the motion of the wheel as a rotation about a moving axis through its center.

180. A wheel 6 in. in diameter rolls on a straight track. At a certain instant its center has a velocity of 3 ft/sec., and an acceleration in the same direction of 2 ft/sec/sec.

Determine the velocity and acceleration of the highest point of the wheel at that instant. Of the lowest point.

181. Suppose the wheel of Prob. 180 to be rolling on the circumference of a cylinder 2 ft. in diameter. At a certain instant, when the wheel is at the top of its path, its center has a velocity (clockwise) of 3 ft/sec. and a tangential acceleration (clockwise) of 2 ft/sec/sec. Determine the velocity and acceleration of the highest point of the wheel at that instant. Of the lowest point. Compare with the results obtained in Prob. 180.

182. The sphere (Fig. 280, p. 174) is suspended from the end of a vertical shaft OA by means of the rod OC extending into and rigidly fastened to the sphere. The shaft and the rod are connected by a Hooke's (flexible) joint. When the shaft is rotated it exerts a torque on the rod which in turn makes the sphere roll around on the cone. Assume that the sphere is 2 ft. in diameter, $R = 4$ ft., $l = 8$ ft., and that the shaft makes 150 rev/min. Determine the angular velocity of the sphere, and the x, y and z components of that velocity.

183. Make a sketch copy of Fig. 280, but omit the cone. Turn the sketch upside down and then represent the cone as *hollow*, and so that the sphere rests *inside* the cone. Take data as in Prob. 182 and determine the angular velocity of the sphere, and the x, y and z components of that velocity.

184. Two men A and B are walking at a speed of 4 mi/hr. along east and west and north and south paths respectively. Compute the velocity of A relative to B when A is walking northward and B eastward; when A is walking northward and B westward.

185. An airplane is flying toward the south at 70 mi/hr., climbing on a path inclined at 10° to the horizontal; the wind is blowing directly north at 20 mi/hr. What are the magnitude and direction of the wind velocity relative to the airplane?

186. At a certain instant a 3 in. shell, still in the gun, has a forward velocity of 700 ft/sec., a forward acceleration of 450,000 ft/sec/sec., an angular velocity of 60 rad/sec., and an angular acceleration of 300,000 rad/sec/sec. (values approximate). Determine the magnitude of the acceleration of a point on the outside surface of the shell at that instant.

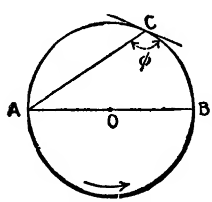

FIG. 78

187. The disk (Fig. 78) is 4 ft. in diameter and is rotating uniformly about O at one rev/sec. A point P is moving uniformly along the diameter AB from A toward B at a speed of 4 ft/sec. Determine the absolute velocity of P when midway between A and O; when midway between O and B.

Suppose that P is moving from C toward A; the angle $\phi = 150°$, and when P reaches A its speed is 6 ft/sec. (along CA). What is the absolute velocity of P then?

188. A certain square is 6 × 6 ft., and its corners are lettered A, B, C and D in succession around the perimeter. The square is rotating uniformly about a line through A perpendicular to its plane at one rev/sec.; a point P is moving uniformly along CD and in that direction at 6 ft/sec. Determine the absolute velocity and acceleration of P when it reaches the mid position between C and D.

CHAPTER XI

189. A (Fig. 79) weighs 200 lbs., B weighs 100 lbs.; the coefficient of friction under A is $\frac{1}{5}$, that under B is $\frac{1}{4}$; $P = 300$ lbs. Determine the acceleration of A and B, and the tension in the rope connecting them.

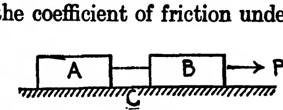

FIG. 79

190. Suppose that the supporting surface in the preceding problem is not horizontal but inclined at 30° to the horizontal. Then solve. *Ans.* $a = 10.07$ *ft/sec/sec.*; *tension* = 197.2 *lbs.*

191. A (Fig. 80) weighs 50 lbs. and B weighs 100 lbs.; the pull P gives A and B an acceleration of 2 ft/sec/sec. Determine the magnitude and direction (referred to the horizontal) of the pressure between A and B.

192. Two bodies are connected somewhat as two cars, and are placed on a plane inclined at 30° to the horizontal. The lower one weighs 600 lbs. and is smooth, that is, there is no resistance to its sliding on the plane. The upper one weighs 1000 lbs., and the coefficient friction under it is $\frac{1}{10}$. With what acceleration will the bodies slide down when released? Will there be tension or pressure at the connection? What is its value?

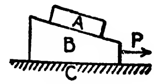

FIG. 80

193. The weights of A, B and C (Fig. 81) are 50, 100 and 200 lbs. respectively. Contacts between A, B and C are very rough; between C and D very smooth; $P = 100$ lbs. Determine the forces which the bodies exert upon each other. Sketch each body separately, showing the forces acting on it.

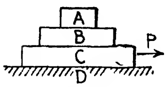

FIG. 81

194. A (Fig. 82) weighs 100 lbs., and B weighs 200 lbs. The coefficient of kinetic friction under B is $\frac{1}{5}$; the coefficient of static friction under A is $\frac{1}{10}$. When $P = 75$ lbs., will A slip? How great is the friction under A? How large a force P would just make A slip?

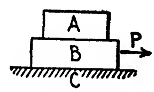

FIG. 82

195. A (Fig. 80) weighs 50 lbs., and B weighs 100 lbs. C is perfectly smooth; the coefficient of static friction between A and B is $\frac{1}{5}$; the angle between the top of B and the horizontal is 25°. How great may P be without making A slip on B? *Ans.* 110.2 *lbs.*

196. Take the weight of the elevator in Prob. 157 as 7500 lbs., and the area of its bottom as 30 ft². Determine the amount and direction of the resultant of all forces acting on the elevator when it was 70 ft. below the top of the air cushion. What forces make up the resultant? Are the records consistent?

197. The following quotation from *Engineering News*, Dec. 3, 1914, relates to tall building elevators: "I and some of my associates have been subjected to retardations as high as 72 ft/sec/sec. without much discomfort. The stress in the ankles is quite noticeable. In air cushion practice it is customary to allow for retardations of five or even six times that of gravity; *i.e.*, retardations up to nearly 200 ft/sec/sec., and it is considered that even this high retardation will not be injurious to life. There are several instances on record where it has been sustained without serious injury." Calculate in the case of a man standing upright in an elevator undergoing a retardation of six times gravity, the pressure on the soles of his feet, and the stress at his neck. (Take the weight of his head as 7 per cent of his total weight.)

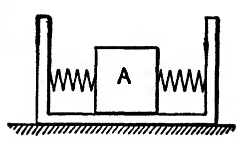

FIG. 83

198. The box shown in Fig. 83 weighs 120 lbs. and the body A 80 lbs.; the system is moving to the right on a floor from an initial impulse. The coefficient of friction between A and the box is zero and that between the box and floor is 0.3. Find the forces on the (like) springs.

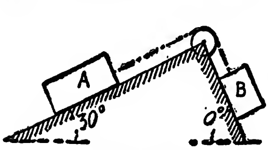

FIG. 84

199. A body weighing 24 lbs. is projected up a 30° incline at a velocity of 20 ft/sec. The coefficient of friction between body and incline is 0.15. Find the position of the body after 5 sec.; after 20 sec.

200. A (Fig. 84) weighs 160 lbs. and B 250 lbs. The coefficient of kinetic friction between A and the incline is 0.3, and between B and the incline 0.25. Determine the acceleration of the system. (Consider the rope very flexible and neglect the mass of rope and pulley.)

201. A, B and C (Fig. 85) weigh 60, 30 and 10 lbs. respectively, and the coefficient of friction between A and D is 0.2. Neglect the stiffness of the rope, its mass, and that of the sheave. Find the tension in the rope between B and C.

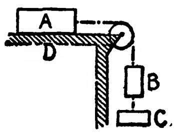

FIG. 85

202. A light flexible rope passes over a frictionless pulley, the mass of which is negligible. To one end of the rope is attached a body A which weighs 100 lbs.; to the other end of the rope is attached a body B which weighs 160 lbs. (a) Determine the acceleration of A and B if the system is allowed to move under the action of gravity. (b) What downward force applied to B would cause an acceleration of 15 ft/sec/sec.? (c) What additional weight, attached to B, would cause an acceleration of 15 ft/sec/sec.?

203. The apparatus in Fig. 237 (page 144) is being used to compress air. The piston is 10 ins. in diameter and weighs 40 lbs.; the lengths of crank and connecting rod are 3 and 10 ins. respectively; the crank rotates at 100 r.p.m.; the pressure on the top of piston is 50 lbs/in². Find the stress in the top of the piston rod when $\theta = 40°$.

204. Referring to the apparatus described in Prob. 166, determine the stress on the specimen at each end of a stroke and at the middle. (Take data as for Prob. 166.)

205. Take data except speed as in preceding problem. Determine the speed which would make the stress on the specimen equal to zero at the upper end of the stroke What would the stress be at the lower end at that speed?

206. A locomotive is running along a level track. The horizontal components of the reactions of the rails on the drivers are forward or back according to certain circumstances; on the other wheels it is opposite to the direction of motion. Call the first force P; the second Q. What is the relation between P and Q (i) when the locomotive speed is constant? (ii) When it is being increased? (iii) When it is being lessened? What does a locomotive do to a bridge over which it is moving in the three cases?

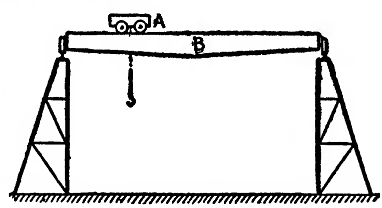

FIG. 86

207. What is the nature of the action of a traveling crane (Fig. 86) on its track (a) when it is starting to "travel" (run down the yard)? (b) When it is starting to "traverse" (crab A runs over the bridge B)? (c) When hoisting of the load begins?

208. Take weights of parts of crane (preceding problem) as follows: bridge B, 52 tons; crab A, 11 tons; load C, 30 tons. Take acceleration of travel 0.7 ft/sec/sec., and acceleration of traverse 0.4 ft/sec/sec.; then calculate the wheel reactions on track so far as possible for commencements of travel and traverse. (Neglect swing of load. What is effect of this error?)

209. A body weighing 10 lbs. falls a distance of 5 ft. upon the top of a spiral spring. The spring is of such stiffness as to be compressed 1 in. by each additional 100 lbs. of load. (a) Determine the velocity of the body after it has compressed the spring 3 ins. (b) Determine the total compression that the body will cause in the spring.

210. At what angle should a smooth chute be inclined in order that objects, allowed to slide down it, shall, within any given time after starting, experience maximum horizontal displacement?

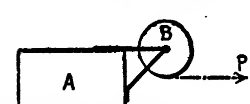

FIG. 87

211. Fig. 87 represents a block A upon which is mounted a drum B. A light rope is wound around the drum so that a horizontal pull P can be applied as shown. The weight of A is 960 lbs., and that of B is 640 lbs. The coefficient of friction under A is 0.3. When $P = 800$ lbs., the acceleration of A has what value?

212. A cylinder C (Fig. 88) is suspended by a cord and rests against a smooth in-

clined plane P as shown. The cylinder weighs 20 lbs.; its diameter is one foot. The plane is rotated at 30 rev/min. about the vertical axis AB. Determine the tension in the cord and the pressure against the plane. *Ans. Tension* = 21.2 *lbs.*

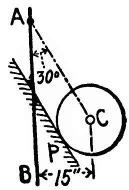

FIG. 88

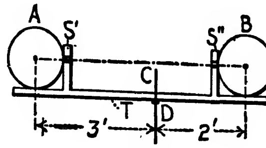

FIG. 89

213. T (Fig. 89) is a horizontal whirling table. A and B are spheres connected by an elastic cord, the tension in which is 30 lbs. when the table is at rest A weighs 10 lbs. and B weighs 40 lbs. What are the pressures of the stops S' and S'' against the spheres when the table is rotated about CD at 20 rev/min.?

214. CD (Fig. 90) is a vertical axis about which E can be rotated. A is a body resting on E, and B is suspended by means of a cord fastened to A as shown. A weighs 10 lbs. and B weighs 20 lbs. Suppose that E makes 30 rev/min.; then compute the pressure at the stop S. The centers of A and B are 5 and 3 ft. from CD respectively. (Neglect friction under A, at B, and the pulley axle.)

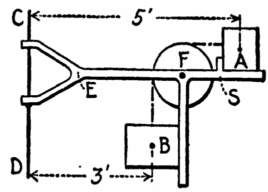

FIG. 90

215. Suppose that A and B in Prob. 214 are rough, the coefficients of static friction being $\frac{1}{4}$ for each. What rate of rotation would lift B?

216. AB (Fig. 91) is a board lying upon a table. C is a vertical peg in the table top projecting upward through a suitable hole in the board. The board weighs 20 lbs. $AC = 8$ ft., $CB = 3$ ft., $DC = 2$ ft. The table top (and board) are spun about C at 400 rev/min. Determine the stress at the smallest section of the board.

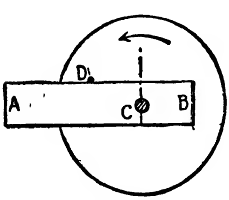

FIG. 91

217. A table with a circular top has a smooth border along the edge and is mounted so that it can be rotated about the vertical through its center. Suppose that a straight rod lies on the top. When the table is rotated the rod rolls to and its ends bear against the border. Take the diameter of the top = 9 ft., length of rod = 6 ft., weight of rod = 10 lbs., and speed of rotation = 200 rev/min. Compute the pressures at the ends of the rod.

218. A disk is so mounted that it can be made to rotate in a plane inclined at 20° to the horizontal. If a small body is placed on the disk, at a distance of 15 ins. from the axis of rotation, what is the maximum uniform speed at which the disk can rotate without causing the body to slip? The coefficient of friction between disk and body is 0.6.

219. The following is an extract from a description of the Sheepshead Bay Motor Racetrack (*Engineering News* for Aug. 19, 1915): "There are two parallel straightaway stretches connected by two turns of 180 deg. each. Each turn consists of a circular arc of about 135 deg. connected by 'spirals' to the straightaway stretches; the radius to the inner edge of the circular track is 850 ft. The outer edges of the circular turns are given a maximum super elevation of 25 ft. 6 in., computed for a speed of 96 mi/hr. by the common railway formula $C = dv^2/gR$. The width d was taken in 14 ft. strips

commencing at the inside of the track, and the super elevation computed for speeds of 40, $52\frac{1}{2}$, 65, $77\frac{1}{2}$ and 96 mi/hr. This gives a cross section theoretically of a parabolic curve." Prove the last statement, and show how the formula gives 25 ft. 6 in.

220. Determine how much the apparent weight of a body (as measured with a spring balance) is diminished by the rotation of the earth at (*a*) the equator and (*b*) latitude 45°. For the purposes of this problem the earth may be assumed to be a sphere with a radius of 4000 miles.

221. (*a*) If the center of gravity of a locomotive driver is not on the axis of the axle, does the pressure which that driver exerts on the track vary during a revolution? (*b*) Does the reaction of the driver on the axle vary?

222. State precisely what answer the "principle of motion of the mass center" enables you to give to each of the following questions:

(*a*) A uniform straight slender rod is set endwise upon a perfectly smooth horizontal floor, in a position not quite vertical. In what manner will the rod fall, and where will it be, relative to its initial position, when it strikes the floor?

(*b*) What if the smooth floor is inclined?

(*c*) The rod lies upon a smooth horizontal floor, its axis pointing north and south. A horizontal force acting due east is applied to the north end of the bar. What can you say as to the resulting motion of the bar?

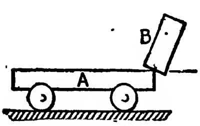

FIG. 92

(*d*) What if a clockwise couple, consisting of two horizontal forces very close together, is applied near the north end of the bar?

(*e*) What if the rod is replaced by a flexible rope or chain?

223. *B* (Fig. 92) is a straight post which rests upon the front edge of the car *A*, at an inclination of 60° to the horizontal. With what acceleration must *A* be made to move along a level track in order that *B* will remain in the position shown. *Ans.* $a = 18.6\ ft/sec/sec.$

224. Assume that the car of the preceding problem moves up a 30° incline with a uniform acceleration of 12 ft/sec/sec. At what angle with the vertical must the post be inclined in order that, as before, it may maintain its position?

225. A homogeneous cylinder weighing 100 lbs. is drawn up an inclined plane by a force of 200 lbs. acting parallel to the plane. The cylinder is 4 ft. long and 2 ft. in diameter; it rests on end with its length normal to the plane. The coefficient of friction between cylinder and plane is 0.2; the angle of inclination of the plane is 30° to the horizontal. Determine the limits between which the point of application of the 200 lb. force must lie in order that the cylinder may not tip over.

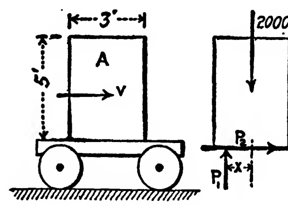

FIG. 93

226. Suppose that the floor of the car and *A* (Fig. 93) are very rough so that *A* will not slip on the car; then ascertain how great an acceleration of the car would result in tipping of *A*.

227. Suppose that the coefficient of friction in Prob. 226 is $\frac{1}{3}$. If the applied push on the car is gradually increased, thus increasing the acceleration gradually, will *A* slip or tip eventually?

228. The Scotch cross-head (Fig. 225, p. 131) presses against the stuffing box and on the cylinder by reason of the weight of the cross-head and the pressure of the crank-pin on it. Suppose that the center of gravity of the cross-head and piston is 15 ins. from the center of the slot, the center of the piston is 24 ins. from the same point, and the center of the stuffing box is 13 ins. from O. The weight of the cross-head and piston is 120 lbs. Determine the pressures mentioned when the steam pressure is 2000 lbs. and the circumstances are as described in Ex. 3, p. 131.

229. "A locomotive weighs 380,000 lbs. Its cab is rigidly mounted on the truck side frames. It is designed to withstand a maximum end bump of 1,000,000 lbs. A

transformer mounted in this locomotive cab weighs 22,000 lbs. and its center of gravity is 28 ins. from its supporting feet. It is to be held down by a total of four ordinary clearance bolts at the lower corners of the transformer tank. The bolt centers will be 40 ins. along the length of the locomotive.

"Assuming a working tensile stress of 9000 lbs. per square inch in the bolts, what size bolts should be used to hold the transformer?" (Westinghouse Electric and Manufacturing Co., Problem Service for Technical Schools.)

230. Answer and discuss the following questions: (*a*) How should a straight uniform rod be set upon a smooth inclined plane in order that it may slide down the plane on end without falling over? (*b*) What if the plane is not smooth? (*c*) A long block or prism stands on end on a smooth horizontal plane. Where should a horizontal pull or push be applied to this block in order to move it along the plane without upsetting it? (*d*) What if the plane is rough? (*e*) If a straight uniform bar is suspended by one end from a cord, and cord and bar are raised to a certain position and allowed to swing freely, as a pendulum, what can you say as to the motion of the bar? (Base your answer wholly on the criterion for motion of translation.) (*f*) What if the cord is attached at the center of the bar? (*g*) Why can an automobile, running at a high speed on the ice, be allowed to skid with little danger of overturning? (*h*) What is the difference, if any, between the reactions on the front and rear wheels of an automobile when it is standing still, running at uniform speed, starting up, and slowing down? (*i*) Answer the same question with respect to a freight car that is drawn along a track, assuming successively that the pull is applied above, at, and below the level of the center of gravity.

231. A, B and C (Fig. 94) weigh 100 lbs., 30 lbs. and 34.4 lbs., respectively. The diameter of C = 2 ft. 3 ins., and the radius of gyration of C about the axis of rotation = 1 ft.; $\phi = 30°$. Friction under A, when the system is moving, = 10 lbs. Determine the acceleration of A, B and C, and the tensions, the system having started without initial velocity. (Neglect axle friction.)

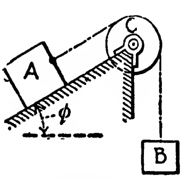

FIG. 94

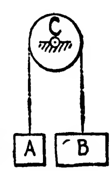

FIG. 95

232. A, B and C (Fig. 95) weigh 50 lbs., 100 lbs. and 150 lbs., respectively. C is a solid disk of cast iron 16 ins. in diameter. Determine the acceleration of A, B, C, and also the pulls of the cord on A and B. (Neglect axle friction.)

Ans. *Acceleration of* $A = 7.18$ *ft/sec/sec.*

233. AB (Fig. 96) is a brake for regulating the descent of the suspended body C. C weighs 1000 lbs., the drum 2000 lbs., the diameter of the drum = 12 ft., that of the brake wheel = 14 ft., a = 4 ft., b = 6 ins., and the radius of gyration of the entire rotating system about the axis of rotation = 4 ft. When P = 100 lbs. and the coefficient of brake friction is $\frac{1}{4}$, what is the acceleration of C? (Neglect axle friction.)

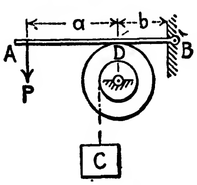

FIG. 96

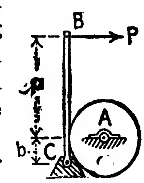

FIG. 97

234. The wheel A (Fig. 97) is a solid cylinder weighing 1000 lbs. and its diameter is 8 ft. It is desired to arrange a brake BC as shown, by means of which the speed of the wheel may be reduced from 100 rev/min. to zero in 10 secs. The coefficient of friction is $\frac{1}{4}$; the available pull P is 100 lbs. Determine the ratio a/b. (Neglect axle friction.)

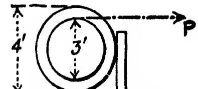

FIG. 98

235. The drum shown in Fig. 98 rests upon a floor and against a low vertical wall as shown. A rope is wound around the axle and passes off horizontally over the wall. The diameter of the drum is 4 ft.; the diameter of the axle is 3 ft.; the radius of gyration of the drum and axle is 20 ins., and their combined weight is 180 lbs. The coefficient

of friction between drum and floor is 0.3 and between drum and wall 0.2. The pull $P = 200$ lbs. Determine the angular acceleration of the drum, and all forces which act upon it.

236. Fig. 99 represents a brake device for regulating the speed at which a load is lowered. It consists of a ring solidly fixed in a horizontal plane, through the center of which passes a vertical shaft. This shaft carries a cross-arm upon which slide two blocks AA. The rope which carries the load passes over a pulley and is wound around the shaft as shown. When the load starts to descend, the shaft rotates and the blocks AA, sliding out to the ends of the cross-arm, bear against the inner side of the ring. The weight of each of the blocks is 64 lbs., the distance from the axis of the shaft to the center of gravity of each block when bearing against the ring is 20 ins., the diameter of the shaft is 6 ins., the coefficient of friction between block and ring is 0.2, and the load W is 1600 lbs. Neglecting the weight of the vertical shaft and the pulley, find the maximum velocity which the load will attain in descending.

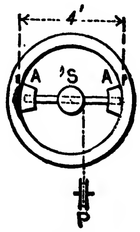

FIG. 99

237. A cylinder is mounted on a horizontal axle so that it can rotate freely. The cylinder weighs 200 lbs., has a diameter of 3 ft., and a radius of gyration of 15 ins. (*a*) A couple of 300 in-lbs., in a vertical plane perpendicular to the axle, acts on the cylinder for one minute. How many revolutions will the cylinder make during that minute and what angular velocity will it have at the end of that time? (*b*) A rope is wound around the cylinder and a force of 50 lbs. applied to it. How long will it take to unwind 100 ft. of the rope?

238. (*a*) What pull applied to the rope (Prob. 237) will give the cylinder an angular acceleration of 3 radians per second per second? (*b*) What weight, attached to the rope and allowed to descend under the action of gravity, would give the cylinder this acceleration? (*c*) What is the greatest acceleration that could possibly be given the cylinder by the latter method?

239. Consider a cylinder of a given moment of inertia, mounted as described in Prob. 237, but with the rope wrapped around the axle instead of around the body of the cylinder. A weight W is attached to the rope and allowed to descend as in Prob. 238. What should be the radius r of the axle in order that the cylinder may be given the maximum angular acceleration?

240. (*a*) A heavy flywheel is mounted on a horizontal shaft that is supported by frictionless bearings. If a rope is wound around this shaft, would it be theoretically possible for a man to climb this rope? (*b*) A disk is mounted on a horizontal axle passing through its center. Where on the disk should a given force be applied in order to cause the maximum angular acceleration? (*c*) A board is mounted in a horizontal position on a vertical axle that passes through its center. To each end of the board is firmly attached a pail. Would a given torque give the system a greater angular acceleration when these pails were filled with water, or when they were filled with some solid material of equal weight?

241. When a slender body, such as a pole, chimney, etc., is tipped over from an upright position, the motion is one of rotation about the point of contact of the body and the surface which supports the body until slip occurs at the point of contact. Assume that the slender body is of uniform section and that it is hinged to the supporting surface so that it cannot slip, and then determine the vertical and horizontal components (V and H) of the supporting force for various positions of the tipping body. Draw curves showing how V and H vary with the angular displacement of the pole from the vertical.

242. Referring to the preceding problem, see that the slender uniform body rests on a horizontal floor and that the coefficient of friction between the body and the floor

is μ. Determine the angle which the body makes with the horizontal at the instant that slipping occurs, assuming the body to have been tipped over from the vertical position. Determine the least value of the coefficient of friction between the body and the floor that would entirely prevent slipping.

243. A heavy spool, consisting of a pair of wheels 3 ft. in diameter attached to either end of a drum 1 ft. in diameter, weighs 200 lbs. and has a radius of gyration with respect to its central axis of 18 in. A rope is wound about the drum, and passes off horizontally from the under side and has applied to it a pull of 60 lbs. If the floor on which the spool rests is so rough as to prevent slipping, determine the acceleration of the drum and the reaction of the floor upon it.

244. A wheel, the center of gravity of which is some distance from its geometrical center, is given a start and allowed to roll freely along a level track. If the track is rough enough to prevent slipping, discuss the motion of the wheel as it rolls along, and the reaction of the floor upon it.

245. Consider a slender uniform rod, suspended by a pin through its upper end, hanging freely. (*a*) Determine the pin reactions at the upper end of the rod at the instant a horizontal force P is applied at the lower end. (*b*) Determine the pin reactions at the instant a horizontal force P is applied at the center of the rod. (*c*) Determine the point at which the horizontal force P should be applied in order that no additional force would be caused to act on the rod at its upper end.

246. Compute the length to the nearest hundredth inch of the simple seconds pendulum for your locality.

247. The length of a simple seconds pendulum at a certain place is 3.56 ft. Find the length of a pendulum which at the same place swings from one side to the other in 5 secs. (U. S. Civil Service examination.)

248. A (Fig. 100) is a rigid piece which can be rotated about the vertical axis BC. D is a vertical bar pinned to A at E, and rests against A at F; the bar is 14 ins. long and weighs 20 lbs. The speed of rotation is 100 rev/min. Determine the pressures on D.

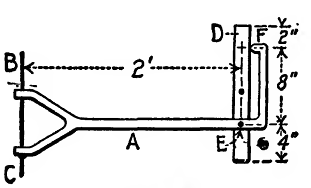

FIG. 100

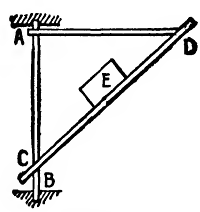

FIG. 101

249. The frame (Fig. 101) can be rotated about the vertical shaft AB. The shaft is 12 ft. long; $AD = 8$ ft., and $BC = 2$ ft. The weights of these members are respectively 500, 200 and 400 lbs. E is 2×4 ft., and perpendicular to paper, 1 ft.; it weighs 300 lbs., and is placed at mid-length of CD. The entire system is rotated at 1800 rev/min. Determine all forces on each member.

CHAPTER XII

250. A block rests on a horizontal floor; the block weighs 100 lbs. and the coefficient of friction between the block and the floor is 0.2. Determine the work done on the block by each of the forces that act on it while it is being dragged a distance of 10 ft. by a force of 60 lbs. that acts: (*a*) Horizontally; (*b*) up at an angle of 30° to the horizontal; (*c*) down at an angle of 30° to the horizontal.

Ans. (*c*) *By* 60 *lb. force,* 519.6 *ft-lbs.; by friction,* 260 *ft-lbs.*

251. A bead moves along a wire bent in the form of a circle of 3 ft. radius, from the lowest to the highest point, while acted on by a vertical force. The magnitude of this vertical force is given by $F = 2\ s$, where F is in pounds and s is the distance in inches, the bead has moved along the wire. Determine the work done by the force during the motion described.

252. (*a*) Does work have to be done in order to exert a force? (*b*) Does work have to be done in order to maintain a force? (*c*) Does work have to be done in order to give a body acceleration? (*d*) Does work have to be done in order to maintain a constant acceleration? (*e*) In what way does the amount of work done by a team in starting a heavy wagon depend on the harness?

253. Consider a small block, drawn across a rough horizontal platform by means of an attached cord. Is any work done on the small block by friction? Consider that the small block is held stationary (as by a string attached to some outside fixed support) and that the platform is drawn out from under it. Is any work done on the block in this case? Suppose both block and platform are moved in the same direction at different speeds? In opposite directions?

254. In order to retard the motion of a launching ship, ropes were fastened to it and to points on the shore, so that the ship broke many of the ropes as it progressed. In order to estimate the retarding effect of each rope broken, tension tests were made on samples of the rope (7-in. manilla). Fig. 102 shows the average tension-stretch

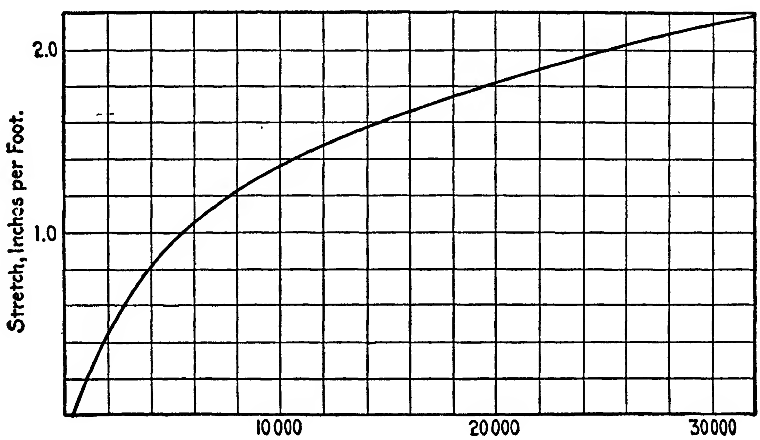

FIG. 102

curve for these tests. The average strength of the samples was about 32,500 lbs. It was assumed that the efficiency of the knots used would be about 80 per cent, and, therefore, that the ropes would fail at about 26,000 lbs. On the basis of this assumption and the curve, it was estimated that each rope (20 ft. long) would do 60,000 ft-lbs. of work on the ship before breaking. Can you check this estimate? (Data taken from *Trans. Soc. Nav. Archts. and Mar. Engrs.*, 1903, p. 295.)

255. A tractor drags a stone-sledge, weighing 2000 lbs., over a hill at uniform speed. The outline of the hill can be obtained by plotting the following data:

Horizontal Dist. ft.	0	100	200	300	400	500	600
Vertical Dist. ft.	0	40	108	164	172	105	80

The coefficient of friction between the ground and the sledge may be taken as 0.8. The tractive force is always parallel to the ground. Determine the work done on the sledge

during the indicated displacement by: (*a*) Gravity; (*b*) friction; (*c*) the tractive force; (*d*) all forces that act on it.

256. C (Fig. 103) is a bead on a circular wire ABD; it is subjected to four forces, F, P, Q and S; $F = 10$ lbs. and is always horizontal; $P = 40$ lbs. and is always directed toward D; Q varies in magnitude and direction so that the simultaneous values of α, β and Q are as follows:

$\alpha =$	0°	15°	30°	45°	60°	75°	90°
$\beta =$	0°	5°	10°	15°	20°	25°	30°
$Q =$	10	15	18	20	15	12	10 lbs.

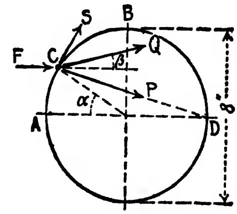

FIG. 103

S is a tangential pull and its value (in pounds) is $40\, s^2$, where s is the arc AC in feet. Compute the amount of work done by each force for the displacement of C from A to B.

257. What power is required to move a block weighing 1200 lbs. up a 30° incline at a uniform speed of 100 ft/sec. if the coefficient of friction between block and plane is 0.2?

258. What horsepower is represented by a continuously flowing jet of water 2 ins. in diameter which has a velocity of 100 ft. per second?

259. What can you say as to the probable magnitude of the power developed when an anvil is struck a violent blow with a hammer? When a rifle is fired? What data would be required for the determination of the average power developed during the blow, or during the discharge, and how would this average power be determined?

260. Fig. 104 represents Durand's dynamometer. A, B, C and D are sprocket wheels of equal diameter; A and B are mounted on a beam XYT which is carried by the well-known Emery steel-plate support or knife-edge at E. The knife-edge rests on the standard R. Sprocket wheels C and D are mounted on R. The bars SS are fastened rigidly to the beam, and engage loosely with a pin on R, thus limiting rotation of the beam. The sprocket chain passes over A, under D, over B, under C, and up to A. The shafts for C and D are extended forward and back; and on these extensions pulleys may be mounted, or universal joint couplings may be attached, for the receipt and delivery of power. (For detailed description see *American Machinist* for June 20, 1907.) $OEO'T$ is horizontal; PQ and IH are vertical; MN and KL are inclined at an angle of 27° with the vertical; $OE = O'E = 12$ ins.; and $ET = 24$ ins. Suppose that an electric motor on the shaft of C turns counterclockwise at 100 rev/min., and transmits to a machine on the shaft of D, and that a weight of 40 lbs. at T keeps the beam XY balanced. What is the power of the motor? *Ans.* 0.402 *hp.*

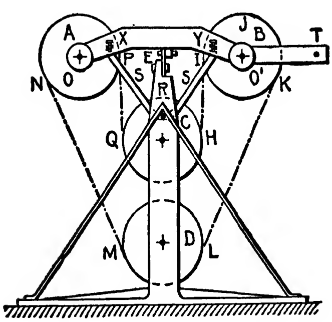

FIG. 104

261. S and S' (Fig. 105) are two portions of a shaft. Arms A and A' are rigidly attached to the adjacent ends of the shaft as shown. The ends of the arms are furnished with hooks which are connected by two like coil springs as shown. Thus it is possible to transmit energy from one portion of the shaft to the other; indeed the device illustrates, in principle, a "transmission dynamometer." Let length of each arm = 12 ins., natural length of each

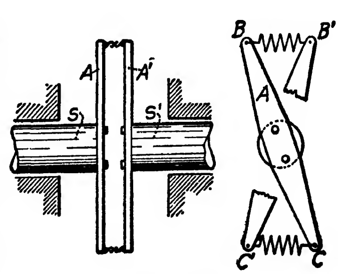

FIG. 105

spring = 8 ins., stiffness of each spring = 40 lbs/in. (40 lbs. pull required for each inch of stretch). When the shaft is rotating at 200 rev/min., the angle between the arms is 60°. What is the horsepower of transmission?

262. Put your solution of the preceding problem into general terms, using the following notation: a = length of each arm in inches, b = natural length of each spring in inches, p = stiffness of spring in pounds per inch, n = speed in revolutions per minute, θ = angle between arms in degrees.

263. When a solid (circular) cylinder is rolling on a straight roadway, what portion of its total kinetic energy is "translational"?

264. Show that the rotational part of the kinetic energy of a rolling sphere is two-sevenths of its total kinetic energy.

265. A certain freight car with its load weighs 60 tons. Each pair of wheels with its axle weighs 1800 lbs., and the radius of gyration of a pair and axle with respect to the axis of the axle is 0.81 ft.; the diameter of the wheels is 33 ins. Determine the ratio of the rotational part of the kinetic energy of the moving car (and load) to the translational part.

266. A certain body weighs 400 lbs., and is dragged along a rough horizontal plane by a force of 80 lbs. The force is inclined 20° upward from the horizontal; the coefficient of friction between the body and plane is about $\frac{1}{10}$. At a certain point in the motion, the velocity of A is 5 ft/sec. What is the velocity of A 10 ft. beyond the point?

Ans. 9.28 *ft/sec.*

267. The suspended body C (Fig. 106) weighs 10 lbs. The coefficient of friction under the brake is $\frac{1}{8}$; $r_1 = 4\frac{1}{2}$ ins., $r_2 = 6$ ins., $a = 2$ ft., and $b = 1$ ft. C is allowed to descend 6 ft., thus turning the wheel, and then the brake is put on, with $P = 20$ lbs. How much farther will C descend? (Neglect axle friction.)

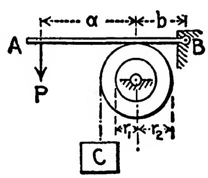

FIG. 106

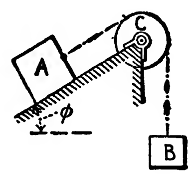

FIG. 107

268. A, B and C, respectively (Fig. 107) weigh 100, 30 and 64.5 lbs. The diameter of C = 30 ins., and its radius of gyration about the axis of rotation = 1 ft.; $\phi = 30°$. The friction under A = 10 lbs. Determine the velocity of the system when A has moved through 10 ft. from rest.

269. A car coasts down a 2 per cent grade, starting from a point 1000 ft. from the bottom. As soon as the level track is reached the brakes are set, locking the wheels. The total weight of the car is 3200 lbs. There are two pairs of wheels, each pair, with the axle, weighing 320 lbs., and having a radius of gyration of 1 ft.; the diameter of the wheels is 3 ft. The total rolling resistance is 10 lbs. and the coefficient of friction between wheels and track is 0.2. Determine how far along the level track the car will go after the brakes are set.

270. A flywheel of a 4 hp. riveting machine fluctuates between 60 and 90 r.p.m. Every two seconds an operation occurs which requires $\frac{7}{8}$ of all the energy supplied for two seconds. Find the moment of inertia of the wheel. (U. S. Civil Service examination.)

271. A particle, under the action of gravity alone, moves from rest from the highest point on the outer surface of a smooth sphere whose diameter is 10 ft. Neglecting friction and the small force necessary to start the particle, find: (a) at what point the

particle leaves the sphere, (b) what the velocity is at the instant of leaving. (U. S. Civil Service examination.)

272. Solve Prob. 209 by the principle of work and kinetic energy.

273. A two hundred yard length of cable is coiled on a drum which is mounted on a horizontal axle at the head of a mine shaft. Fifty feet of the cable is allowed to run out, the drum is brought to rest momentarily, and is then released and the remainder of the cable allowed to run out freely. The cable weighs 0.6 lb. per ft. and may be regarded as perfectly flexible; the drum is 3 ft. in diameter, weighs 200 lbs., and has a radius of gyration of 14 ins.; the friction on the axle bearings is equivalent to a constant resisting torque of 15 ft-lb. Determine the rate at which the drum is rotating at the instant the cable has become completely unwound.

274. A 3 in. rifle has a barrel 6.2 ft. long and shoots a shell weighing 15 lbs.; the cross section area of the bore is 7.28 sq. ins. The powder pressure may be taken as varying linearly with the displacement of the projectile, from zero up to a maximum of 33000 lbs/sq. in. when the projectile has moved 0.5 ft., then to zero when the projectile is at the muzzle. (This does not closely approximate the actual way in which the pressure varies, but the assumption will serve the present purpose.) Neglect the effect of recoil and friction, and determine the muzzle velocity of the projectile and the maximum power developed during the discharge.

275. If a freight car is given a certain velocity and then left to coast along a perfectly smooth level track, will its speed change when it passes from a straight portion of the track to a curved portion, and vice-versa? Explain your answer (a) on the principle of work and energy and (b) by the principle of force and acceleration.

276. A (Fig. 108) is a long board supported in a horizontal position; B is a block provided with end guides so that it can be slid along the board without turning; and C is a heavy piece that can be slid across the board. Suppose that B—and with it C—is moved to the left with a velocity of 4 ft/sec., and C is moved across the board, in the direction indicated, with a velocity of 0.2 ft/sec. (a) What is the direction of the friction exerted by A on C? (b) What is the amount of that force if the weight of C—all supported by A—is 10 lbs., and the coefficient of kinetic friction between A and B is 0.4? (c) If the coefficient between B and C is 0.4, what is the amount of friction between them? (d) Compare the total friction on C just found, with its value when C is slid across but not also along the board. (e) Make a similar comparison on the supposition that the contact between B and C is frictionless.

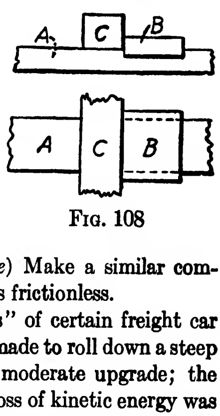

FIG. 108

277. For the purposes of comparing the "running qualities" of certain freight car trucks, they were tested substantially as follows: Each one was made to roll down a steep incline to give it "initial velocity," and then it passed onto a moderate upgrade; the velocity was measured at two points on the upgrade; then the loss of kinetic energy was computed. These losses furnished a comparison. The upgrade was 0.38 per cent, and the points at which velocities were measured were 257.2 ft. apart. One of these trucks (four-wheeled) weighed 18,150 lbs.; each pair of wheels and axle 1800 lbs. The diameter of wheels was 33 ins.; the radius of gyration of a pair and axle was 0.81 ft. In one test the velocities at the two points were 14.95 and 11.05 ft/sec. Determine the average "truck resistance," a single imaginary force equivalent to actual resistances, not including gravity, on the truck. (Experiments by Prof. L. E. Endsley for American Steel Foundries.) *Ans.* 50 *lbs.*

278. Find the total work done at the drawbar of a locomotive in starting a 200-ton train against a one per cent grade from rest to 30 mi/hr. in 300 ft. when the frictional resistances are 10 lbs/ton (assumed the same for all speeds). (Mass. Civil Service examination.)

279. In Fig. 168, page 96, the load $W = 18{,}000$ lbs.; the diameter of the rollers = 15 ins.; the coefficient of rolling resistance "under" the rollers = 0.020, that "over" the rollers = 0.025. How large a force P is required to move the load? Determine the two forces which act upon a roller supposing that the load is distributed equally among the rollers.

280. "To easily push a pulley onto a snugly fitting arbor or shaft, one turns the pulley while pushing." Explain how the turning lessens the necessary push.

281. To turn a certain pulley on a snugly fitting shaft, 3 inches in diameter, requires a torque of 40 ft.-lbs. How large a pull is required to strip the pulley from its shaft at a speed v when the wheel is being turned so that the rate of circumferential slip is V?

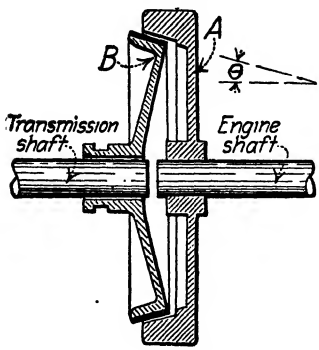

FIG. 109

282. Fig. 109 represents a cone friction clutch, used in some automobiles. A is the engine flywheel; the inner face of the rim is conical. B is the other cone keyed to the transmission shaft but so that B can be slid along the shaft, toward or from the cone A. Generally, the clutch is "thrown in" (B pressed against A) when the engine (and flywheel) are running. The cones slip over each other, the (kinetic) friction on B increasing the speed of B until its speed reaches that of A. Thereafter the friction is static. Suppose that the engine can deliver a torque T at the flywheel at a certain speed of running. To transmit that torque the clutch must develop a circumferential friction, $T \div \frac{1}{2}d$; and hence a normal pressure $2T \div \mu d$. Show that the force required for pressing the cones together is between

$$2T\sin\theta \div \mu d \quad \text{and} \quad 2T(\sin\theta + \mu\cos\theta) \div \mu d.$$

283. Fig. 110 shows plan of a capstan and hauling tackle, and an elevation of the barrel of the capstan. The sweep (arm) to which the horse is hitched at A is 11 ft. long; there is ne sheave in each block B and C. Assume that the horse can exert a prolonged, steady pull of 200 lbs., and make an estimate (supported by calculation) of the pull which can be exerted at the load. (Neglect friction on the barrel.)

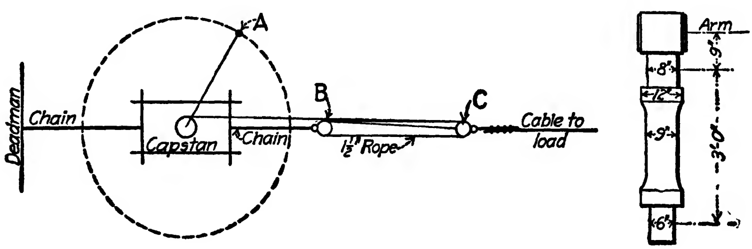

FIG. 110

284. Fig. 111 represents the arrangement of tackle, engines, etc., used for moving a large building (three stories, 120 × 142 ft., weighing about 8000 tons). Pulls were applied at six points on the rear of the building as shown. The four blocks under and the three immediately in front of the building are single (one sheave or pulley in each); A and B are single, C and D double, and E and F triple. The pulling cable from each engine extends to A, and is reeved through A and B, ending at A; a second cable is fastened to A and reeved through C and D, ending at C; a third cable is fastened to C and reeved through E and F, ending at E. Blocks E are merely hooked to the three

blocks immediately in front of the building; blocks B, D and F are held in place by cables fastened to deadmen (buried logs or the like). The runs of cable from A to C and from C to E are really parallel to the main runs; they are shown inclined to avoid confusion of lines. The pull of each engine was about one ton. (For fuller description see

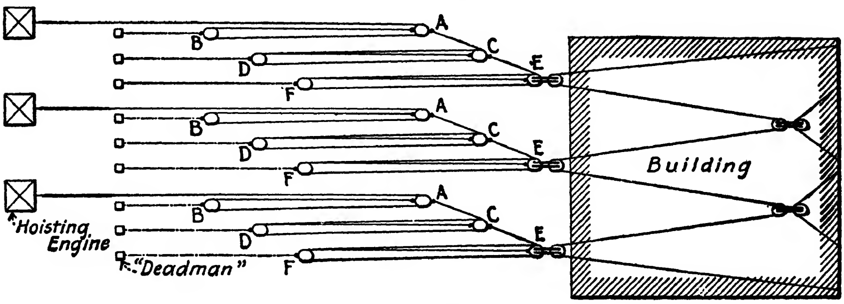

FIG. 111

Engineering Record for Nov. 22, 1913.) Assume K to be 1.15, and compute the total pull exerted on the building; also the pull exerted on each deadman cable. Which one (or ones) of all the cables is subjected to the greatest pull?

285. Fig. 112 represents the mechanism for operating a small bascule bridge of a single draw span. The train of gears, A, B, C, D and E rests on the (fixed) approach span. The quadrant and the draw span are keyed to the same trunnion, supported on the pier shown. When the hand crank is turned counterclockwise (in the view shown),

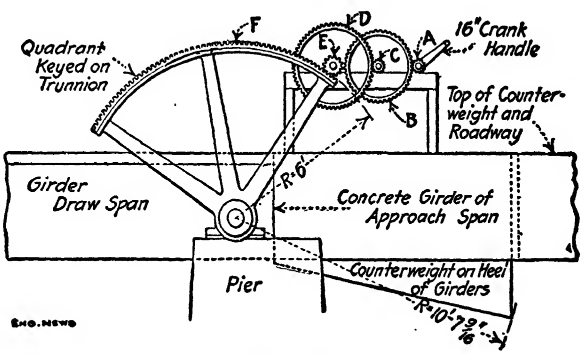

FIG. 112

the quadrant rotates clockwise, and the free end of the draw span lifts. The total weight of the draw span and counterweight is 115,000 lbs., and the center of gravity of that (moving) part of the bridge is in the axis of the trunnion. The trunnion is 7 ins. in diameter. The following description of the gear train is sufficient for our purpose:

Gear	A	B	C	D	E	Quadrant
Number of teeth........	13	94	15	122	11	57

(For fuller description see *Engineering News* for July 24, 1913, or the paper by the designer, Prof. L. E. Moore, in *Engineering and Contracting* for Aug. 13, 1913.) Determine how large a force applied to the crank handle at right angles to the crank is required to raise the draw span.

CHAPTER XIII

286. Water is flowing through a certain 6-in. pipe at a velocity of 4 ft/sec. Compute the resultant pressure of the water against a right-angle bend in the pipe. (Assume that the water pressure is the same at both ends of the bend, and equals 100 lbs/in².)

287. Actually, the water pressure (referring to the preceding problem) is greater at the inlet end of the bend. Assume that the pressures are 104 and 100 lbs/in²; then solve.

288. A certain three and one-half inch hose is conducting water at a velocity of 20 ft/sec. There is a circular bend of 180° in the hose; the radius of the bend is 8 ft. Assume water pressure at both ends of the bend to be 100 lbs/in². Determine the resultant water pressure on the bend. How much pressure (tending to straighten the hose) is there per inch of bend.

289. A body weighing 800 lbs. is dragged along a smooth floor by a horizontal force which varies uniformly with the displacement, the force being zero when the displacement = 0 and 40 lbs. when the displacement = 10 ft. Initial velocity (when $s = 0$) is 2 ft/sec. Determine the time-average value of the force for the 10 ft.

290. Fig. 113 is a part copy of a figure from a report on certain tests of an hydraulic (railway) buffer by Mr. Carl Schwartz, published in the *Journal of the American Society*

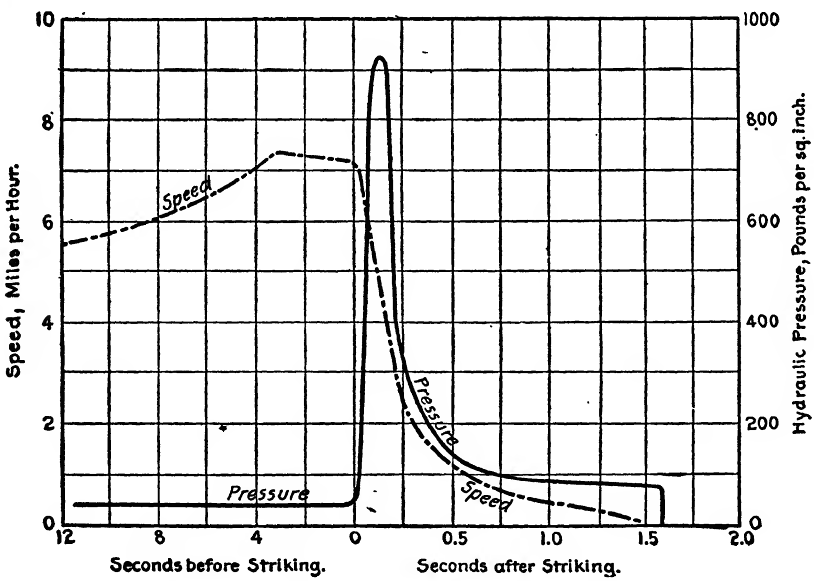

Fig. 113

of Mechanical Engineers for June, 1913. An abstract of the report is printed in *Engineering News* for Sept. 11, 1913. The buffer consists essentially of a cylinder 22 ins. in diameter, and a piston; the working stroke is 11 ft. The buffer is firmly anchored at the stopping point, with the piston rod in the line of approach of the buffer of the car or locomotive to be stopped. The cylinder is grooved so as to allow water to pass by the piston during a stop.

The curve marked "speed" shows how the speed of the locomotive, in this instance, varied during the 12 secs. preceding impact, and also during the impact. Thus the speed was about 5.6 mi/hr. at the beginning of the test; it increased to about 7.3 in $8\frac{1}{2}$ secs.; then it decreased uniformly up to the instant of impact after which it decreased

much more rapidly. The curve marked pressure shows how the hydraulic pressure behind the piston varied during the impact. Thus the initial pressure on each side of the piston was about 45 lbs/in^2; after the instant of impact the pressure shot up to a maximum of 925 lbs/in^2, and then decreased to about 80. The entire travel of the piston in this case was 3 ft. (not indicated in the figure). The locomotive weighed 100 tons.

Compute the time-average and the space-average force which stopped the locomotive, neglecting the effect of the so-called train resistance. Estimate the train resistance from the retardation of the locomotive just before the impact, and then recompute the averages just mentioned. Measure the area under the pressure curve and interpret it. Does the shape of the curve suggest any improvement in the buffer?

291. The power of an operating hydraulic turbine equals the product of the angular velocity of the turbine and the rate at which angular momentum (about the axis of rotation) of the flowing water is changed in its passage through the turbine. Prove.

CHAPTER XIV

292. Describe the gyrostatic reaction which a screw-propelled ship sustains when pitching (in a rough sea).

293. In the *General Electric Review*, Vol. IX, pages 117 and 118, there appears the following: "The spin of a precessing body increases the centrifugal force about the axis of precession. Take the case of a wheel spinning about a horizontal axis supported at one end which is precessing about a vertical axis through the point of support. The total centrifugal force is

$$\frac{WV^2}{gR}\left(1 + \frac{k^2p^2}{2\,r^2}\right)$$

which equals the ordinary centrifugal force WV^2/gR plus the additional centrifugal force due to spin (gyroscopic centrifugal force) $(WV^2k^2p^2)/(gR\,2\,r^2)$. W = weight of the gyroscope, k = its radius of gyration, R = the radius of the circle of precession, r = the radius of the spinning wheel, V = the linear velocity of the precession, v = the peripheral velocity of the wheel, and p = the ratio v/V." Presumably, R means the radius of the circle described by the mass-center of the wheel. Ascertain in your own way whether any force, appropriately called centrifugal force, has the value above stated in the case in question.

294. On page 144 of the journal mentioned in the preceding problem there appears this statement. "The total vertical force on the outside rail [car wheels running around a curve] due to gyroscopic action will therefore be $(3\ WV^2k^2) \div (2\ gRrx)$." W = the weight of a pair of wheels and axle (presumably), k = radius of gyration of the pair and axle (about their axis), r = the radius of the wheels, R = radius of the curve, x = gage of the track, and V = the velocity of the car. Can you prove the statement?

295. Fig. 114 represents in principle the Griffin Mill for grinding cement. The cross piece of the (upright) frame supports the upper (vertical) shaft S by means of a thrust ball bearing. The large pulley P is rigidly fastened to the shaft. The pulley hub HH is extended downward and is restrained laterally by the guides GG, thus virtually forming an extension of the shaft. The "roll" is rigidly fastened to the "roll shaft" which is driven by the pulley P through the equivalent of a universal joint connection. The "die" is a hard metal ring between which and the roll the grinding of the cement takes place as explained presently. When the mill is idle, the roll shaft hangs in a vertical position; if the pulley be rotated the roll shaft rotates in the vertical position with the pulley. When it is desired to start the mill for grinding, the roll is

first pulled outward "with an iron hook," and then the power is turned on at the pulley. The roll shaft rotates with the pulley; promptly, the roll begins and continues to roll on the die (ring), a great pressure being developed between roll and die. Material to be ground is fed into the mill so that some is caught between the roll and the die and then pulverized. Suitable paddles on the lower side of the roll continually toss the material which collects in the recess of the base; eventually it is caught between roll and die.

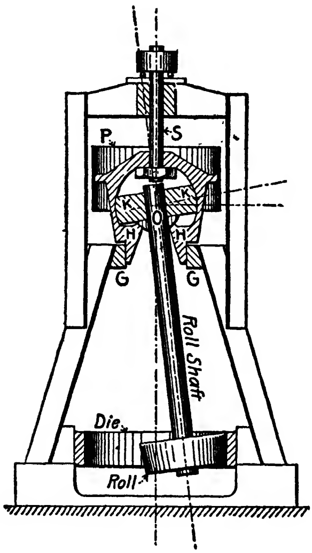

FIG. 114

It will be noted that the roll and its shaft constitute a large gyrostat. We now propose the problem of determining the pressure between the roll and the ring when the mill is operating. The makers (Bradley Pulverizer Co.) state it to be about 15,000 lbs. for their giant size when run at a pulley speed of 165 to 170 rev/min. The following data, approximated in some cases, was taken from drawings furnished by the makers of the mill. The die is 40 ins. in diameter (inside), 8 ins. high; from the plane of its top to the point of suspension O is 5 ft. $4\frac{1}{2}$ ins. The roll weighs 880 lbs.; its larger diameter is 24 ins. The roll shaft weighs 600 lbs.; its length over all is 6 ft. $9\frac{1}{2}$ ins.; its point of suspension O is 6 ins. from the upper end; its diameter varies from $5\frac{1}{8}$ ins. at the cross-head to $6\frac{1}{2}$ ins. at the roll but the ends in the cross-head and roll are tapered. For simplicity, make the following approximations: roll-shaft uniform diameter is $5\frac{3}{4}$ ins., smaller diameter of roll = 22 ins., and its thickness is 8 ins. As a further close approximation for locating center of gravity and determining required moments of inertia, assume that the roll is a cylinder 23 ins. in diameter and 8 ins. thick (with $5\frac{3}{4}$ in. hole for the roll shaft).

APPENDIX B

296. Show that the moment of inertia of the slender wire AB (Fig. 115) about the x-axis is $\frac{1}{2}\,Mr^2[1 - (\sin\alpha\cos\alpha)/\alpha]$, where M = mass of the wire.

297. Show that the moment of inertia of a right circular cone about its axis is $\frac{3}{10}\,Mr^2$, where M = the mass of the cone and r = the radius of its base.

298. Show that the moment of inertia of the ring or torus (Fig. 116) about the z-axis is $M(R^2 + \frac{3}{4}\,r^2)$, where M = the mass of the ring.

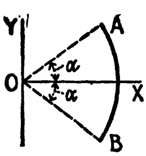

FIG. 115

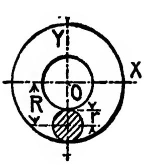

FIG. 116

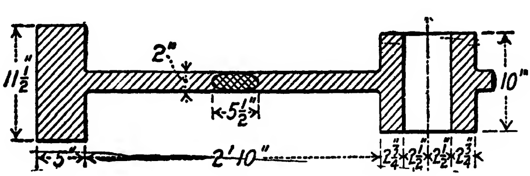

FIG. 117

299. The length of a homogeneous right elliptic prism is l, and the semi-axes of its cross section are a and b. Prove that the radius of gyration of the prism with respect to a line through its center of gravity parallel to the axis b is $(\frac{1}{4}\,a^2 + \frac{1}{12}\,l^2)^{\frac{1}{2}}$.

300. Fig. 117 is a section of a cast-iron flywheel; there are six spokes. The cross

section of each spoke is elliptical, the axes of the ellipse being 2 ins. and 5½ ins. long. Compute the moments of inertia of rim, spokes and hub, with respect to the axis of the wheel; also the radius of gyration of the wheel about that axis.

301. A solid piece of cast iron consists of a right circular cylinder 4 ft. in diameter and 10 ft. long, and a right circular cone 4 ft. in diameter and 7 ft. long, placed end to end. Determine the moment of inertia and radius of gyration of the body with respect to the common axis of cone and cylinder.

302. The moment of inertia of a sphere with respect to a diameter is given by $\frac{2}{5} Mr^2$. What is the moment of inertia of a cast-iron sphere, 24 ins. in diameter, with respect to a tangent line?

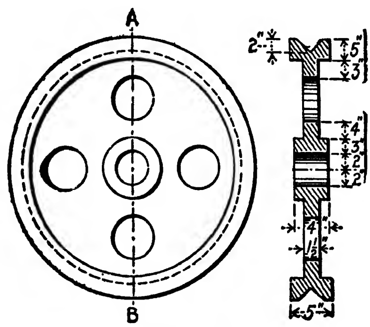

Fig. 118

303. Determine the moment of inertia and radius of gyration of the cast-iron pulley represented in Fig. 118 with respect to its own axis. The width of the groove in the rim is 3 ins.; the diameter of each of the four holes in the flat part of the wheel is 6 ins.

304. A certain right cone with a circular base is homogeneous; the diameter of its base is 4 ft.; the altitude is 6 ft.; and half the apex angle is 20°. Determine the radius of gyration of the cone with respect to an element of its curved surface.

SINES, angles 0° to 45°. Example, sin 33.3° = 0.5490

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											0.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	87	17
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	86	17
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0.0872	85	17
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	84	17
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	83	17
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	82	17
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	81	17
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	0.1736	80°	17
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79	17
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78	17
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77	17
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76	17
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	0.2588	75	17
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74	17
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73	17
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72	17
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71	17
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	0.3420	70°	16
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69	16
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68	16
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67	16
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66	16
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	0.4226	65	16
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64	16
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63	16
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62	16
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61	15
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	0.5000	60°	15
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59	15
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58	15
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57	15
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56	15
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	0.5736	55	14
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54	14
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53	14
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52	14
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51	14
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	0.6428	50°	13
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49	13
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48	13
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47	13
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46	13
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	0.7071	45°	12
45°	0.7071												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSINES, angles 45° to 90°. Example, cos 66.6° = 0.3971

SINES, angles 45° to 90°. Example, sin 66.6° = 0.9178

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											0.7071	**45°**	
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	**44**	12
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	**43**	12
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	**42**	12
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	**41**	12
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	0.7660	**40°**	11
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	7771	**39**	11
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	**38**	11
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	**37**	11
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	**36**	10
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	0.8192	**35**	10
55	0.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	**34**	10
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	**33**	10
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	**32**	9
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	**31**	9
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	0.8660	**30°**	9
60°	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	**29**	9
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	**28**	8
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	**27**	8
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	**26**	8
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	0.9063	**25**	7
65	0.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	**24**	7
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	**23**	7
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	**22**	7
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	**21**	6
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	0.9397	**20°**	6
70°	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	**19**	6
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	**18**	6
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	**17**	5
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	**16**	5
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	0.9659	**15**	5
75	0.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	**14**	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	**13**	4
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	**12**	4
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	**11**	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0.9848	**10°**	3
80°	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	**9**	3
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	**8**	3
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	**7**	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	**6**	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	.9962	**5**	2
85	0.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	**4**	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	**3**	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	**2**	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0.9998	**1**	0
89	0.9998	9999	9999	9999	9999	0000	0000	0000	0000	0000	1.0000	**0°**	0
90°	1.0000												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSINES, angles 0° to 45°. Example, cos 33.3° = 0.8358

TANGENTS, angles 0° to 45°. Example, tan 33.3° = 0.6569

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°9			Avge. diff.
											.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	87	17
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	86	18
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0.0875	85	18
5	0.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	84	18
6	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	83	18
7	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	82	18
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	81	18
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	0.1763	80°	18
10°	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	79	18
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	78	18
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	77	18
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	76	18
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	0.2679	75	19
15	0.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	74	19
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	73	19
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	72	19
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	71	19
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	0.3640	70°	20
20°	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	69	20
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	68	20
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	67	21
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	66	21
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	0.4663	65	21
25	0.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	64	21
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	63	22
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	62	22
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	61	23
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	0.5774	60°	23
30°	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	59	24
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	58	24
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	57	25
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	56	25
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	0.7002	55	26
35	0.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	54	26
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	53	27
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	52	28
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	51	28
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	0.8391	50°	29
40°	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	49	30
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	48	31
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47	32
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	46	33
44	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	45°	34
45°	1.0000												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COTANGENTS, angles 45° to 90°. Example, cot 66.6° = 0.4327

TANGENTS, angles 45° to 90°. Example, tan 66 6°=2.311

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											1.0000	**45°**	
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	**44**	35
46	0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	**43**	37
47	0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	**42**	38
48	1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	**41**	40
49	1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1.1918	**40°**	41
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	**39**	43
51	2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	**38**	45
52	2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	**37**	47
53	3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	**36**	49
54	3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	1.4281	**35**	52
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	**34**	55
56	4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	**33**	57
57	5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	**32**	60
58	6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	**31**	64
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	1.7321	**30°**	67
60°	1.732	1.739	1.746	1.753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	**29**	7
61	1.804	1.811	1.819	1.827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	**28**	8
62	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	**27**	8
63	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	**26**	9
64	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	**25**	9
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	**24**	10
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	**23**	11
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	**22**	12
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	**21**	13
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	**20°**	14
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	**19**	16
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	**18**	17
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	**17**	19
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	**16**	22
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	**15**	24
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	**14**	28
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	**13**	32
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	**12**	37
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	**11**	44
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	**10°**	53
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	**9**	
81	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	**8**	
82	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	**7**	
83	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	**6**	
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20	11.43	**5**	
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95	14.30	**4**	
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46	19.08	**3**	
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	**2**	
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	**1**	
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	**0°**	
90°	∞												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COTANGENTS, angles 0° to 45°. Example, cot 33.3°=1.5224

SECANTS, angles 0° to 45°. Example, sec 33.3° = 1.1964

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											1.0000	90°	
0°	1.0000	0000	0000	0000	0000	0000	0001	0001	0001	0001	0002	89	0
1	0002	0002	0002	0003	0003	0003	0004	0004	0005	0006	0006	88	0
2	0006	0007	0007	0008	0009	0010	0010	0011	0012	0013	0014	87	1
3	0014	0015	0016	0017	0018	0019	0020	0021	0022	0023	0024	86	1
4	0024	0026	0027	0028	0030	0031	0032	0034	0035	0037	1.0038	85	1
5	1.0038	0040	0041	0043	0045	0046	0048	0050	0051	0053	0055	84	2
6	0055	0057	0059	0061	0063	0065	0067	0069	0071	0073	0075	83	2
7	0075	0077	0079	0082	0084	0086	0089	0091	0093	0096	0098	82	2
8	0098	0101	0103	0106	0108	0111	0114	0116	0119	0122	0125	81	3
9	0125	0127	0130	0133	0136	0139	0142	0145	0148	0151	1.0154	80°	3
10°	1.0154	0157	0161	0164	0167	0170	0174	0177	0180	0184	0187	79	3
11	0187	0191	0194	0198	0201	0205	0209	0212	0216	0220	0223	78	4
12	0223	0227	0231	0235	0239	0243	0247	0251	0255	0259	0263	77	4
13	0263	0267	0271	0276	0280	0284	0288	0293	0297	0302	0306	76	4
14	0306	0311	0315	0320	0324	0329	0334	0338	0343	0348	1.0353	75	5
15	1.0353	0358	0363	0367	0372	0377	0382	0388	0393	0398	0403	74	5
16	0403	0408	0413	0419	0424	0429	0435	0440	0446	0451	0457	73	5
17	0457	0463	0468	0474	0480	0485	0491	0497	0503	0509	0515	72	6
18	0515	0521	0527	0533	0539	0545	0551	0557	0564	0570	0576	71	6
19	0576	0583	0589	0595	0602	0608	0615	0622	0628	0635	1.0642	70°	7
20°	1.0642	0649	0655	0662	0669	0676	0683	0690	0697	0704	0711	69	7
21	0711	0719	0726	0733	0740	0748	0755	0763	0770	0778	0785	68	7
22	0785	0793	0801	0808	0816	0824	0832	0840	0848	0856	0864	67	8
23	0864	0872	0880	0888	0896	0904	0913	0921	0929	0938	0946	66	8
24	0946	0955	0963	0972	0981	0989	0998	1007	1016	1025	1.1034	65	9
25	1.1034	1043	1052	1061	1070	1079	1089	1098	1107	1117	1126	64	9
26	1126	1136	1145	1155	1164	1174	1184	1194	1203	1213	1223	63	10
27	1223	1233	1243	1253	1264	1274	1284	1294	1305	1315	1326	62	10
28	1326	1336	1347	1357	1368	1379	1390	1401	1412	1423	1434	61	11
29	1434	1445	1456	1467	1478	1490	1501	1512	1524	1535	1.1547	60°	11
30°	1.1547	1559	1570	1582	1594	1606	1618	1630	1642	1654	1666	59	12
31	1666	1679	1691	1703	1716	1728	1741	1753	1766	1779	1792	58	13
32	1792	1805	1818	1831	1844	1857	1870	1883	1897	1910	1924	57	13
33	1924	1937	1951	1964	1978	1992	2006	2020	2034	2048	2062	56	14
34	2062	2076	2091	2105	2120	2134	2149	2163	2178	2193	1.2208	55	15
35	1.2208	2223	2238	2253	2268	2283	2299	2314	2329	2345	2361	54	15
36	2361	2376	2392	2408	2424	2440	2456	2472	2489	2505	2521	53	16
37	2521	2538	2554	2571	2588	2605	2622	2639	2656	2673	2690	52	17
38	2690	2708	2725	2742	2760	2778	2796	2813	2831	2849	2868	51	18
39	2868	2886	2904	2923	2941	2960	2978	2997	3016	3035	1.3054	50°	19
40°	1.3054	3073	3093	3112	3131	3151	3171	3190	3210	3230	3250	49	20
41	3250	3270	3291	3311	3331	3352	3373	3393	3414	3435	3456	48	21
42	3456	3478	3499	3520	3542	3563	3585	3607	3629	3651	3673	47	22
43	3673	3696	3718	3741	3763	3786	3809	3832	3855	3878	3902	46	23
44	3902	3925	3949	3972	3996	4020	4044	4069	4093	4118	1.4142	45°	24
45°	1.4142												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSECANTS, angles 45° to 90°. Example, csc 66.6° = 1.0896

SECANTS, angles 45° to 90°. Example, sec 66.6° = 2.518

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											1.4142	45°	
45°	1.4142	4167	4192	4217	4242	4267	4293	4318	4344	4370	4396	44	25
46	4396	4422	4448	4474	4501	4527	4554	4581	4608	4635	4663	43	27
47	4663	4690	4718	4746	4774	4802	4830	4859	4887	4916	4945	42	28
48	4945	4974	5003	5032	5062	5092	5121	5151	5182	5212	5243	41	30
49	5243	5273	5304	5335	5366	5398	5429	5461	5493	5525	1.5557	40°	31
50°	1.5557	5590	5622	5655	5688	5721	5755	5788	5822	5856	5890	39	33
51	5890	5925	5959	5994	6029	6064	6099	6135	6171	6207	6243	38	35
52	6243	6279	6316	6353	6390	6427	6464	6502	6540	6578	6616	37	37
53	6616	6655	6694	6733	6772	6812	6852	6892	6932	6972	7013	36	40
54	7013	7054	7095	7137	7179	7221	7263	7305	7348	7391	1.7434	35	42
55	1.7434	7478	7522	7566	7610	7655	7700	7745	7791	7837	7883	34	45
56	7883	7929	7976	8023	8070	8118	8166	8214	8263	8312	8361	33	48
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58	8871	8924	8977	9031	9084	9139	9194	9249	9304	9360	1.9416	31	54
59	1.9416	9473	9530	9587	9645	9703	9762	9821	9880	9940	2.0000	30°	58
60°	2.000	2.006	2.012	2.018	2.025	2.031	2.037	2.043	2.050	2.056	2.063	29	6
61	2.063	2.069	2.076	2.082	2.089	2.096	2.103	2.109	2.116	2.123	2.130	28	7
62	2.130	2.137	2.144	2.151	2.158	2.166	2.173	2.180	2.188	2.195	2.203	27	7
63	2.203	2.210	2.218	2.226	2.233	2.241	2.249	2.257	2.265	2.273	2.281	26	8
64	2.281	2.289	2.298	2.306	2.314	2.323	2.331	2.340	2.349	2.357	2.366	25	8
65	2.366	2.375	2.384	2.393	2.402	2.411	2.421	2.430	2.439	2.449	2.459	24	9
66	2.459	2.468	2.478	2.488	2.498	2.508	2.518	2.528	2.538	2.549	2.559	23	10
67	2.559	2.570	2.581	2.591	2.602	2.613	2.624	2.635	2.647	2.658	2.669	22	11
68	2.669	2.681	2.693	2.705	2.716	2.729	2.741	2.753	2.765	2.778	2.790	21	12
69	2.790	2.803	2.816	2.829	2.842	2.855	2.869	2.882	2.896	2.910	2.924	20°	13
70°	2.924	2.938	2.952	2.967	2.981	2.996	3.001	3.026	3.041	3.056	3.072	19	15
71	3.072	3.087	3.103	3.119	3.135	3.152	3.168	3.185	3.202	3.219	3.236	18	16
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80°	5.759	5.816	5.875	5.935	5.996	6.059	6.123	6.188	6.255	6.323	6.392	9	
81	6.392	6.464	6.537	6.611	6.687	6.765	6.845	6.927	7.011	7.097	7.185	8	
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83	8.206	8.324	8.446	8.571	8.700	8.834	8.971	9.113	9.259	9.411	9.567	6	
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87	19.11	19.77	20.47	21.23	22.04	22.93	23.88	24.92	26.05	27.29	28.65	2	
88	28.65	30.16	31.84	33.71	35.81	38.20	40.93	44.08	47.75	52.09	57.30	1	
89	57.30	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	0°	
90°	∞												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSECANTS, angles 0° to 45°. Example, csc 33.3° = 1.8214

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